7.5.15 Assume $\{f_n\}_{n\in\mathbb{N}} \subseteq \overline{\mathcal{M}}^+$, $f \in \overline{\mathcal{M}}^+$,
$f_n \to f$ pointwise on X, $f_n \leq f$ for all $n \in \mathbb{N}$.
Then $\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$:

Since $f = \liminf_{n\to\infty} f_n$ (pt. wise), we get from Fatou's lemma that

$$\int f \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu \leq \limsup_{n\to\infty} \int f_n \, d\mu \quad . \ (*)$$

Moreover, since $f_n \leq f$, we have $\int f_n \, d\mu \leq \int f \, d\mu$ for <u>all</u> n.
Thus $\limsup_{n\to\infty} \int f_n \, d\mu \leq \int f \, d\mu \ (**)$

Combining $(*)$ and $(**)$, we see that

$$\int f \, d\mu = \liminf_{n\to\infty} \int f_n d\mu = \limsup_{n\to\infty} \int f_n d\mu, \text{ i.e.}$$

$$\int f \, d\mu = \lim_{n\to\infty} \int f d\mu .$$

7.6.9 Let μ be the Leb. meas. on $\mathbb{R}$, $a, b \in \mathbb{R}$, $a < b$.
$f : [a,b] \to \mathbb{R}$ bounded and Riemann-integrable.
Then f is Leb. integrable on $[a,b]$ and
$$\int_{[a,b]} f \, d\mu = \int_a^b f(x)\,dx.$$

Let $M > 0$ be such that $|f| \leq M$ on $[a,b]$, i.e.
$-M \leq f(x) \leq M$ for all $x \in [a,b]$. So we
set $g(x) = f(x) + M$ for $x \in [a,b]$, and set that
$0 \leq g \leq 2M$ on $[a,b]$. Moreover, g is clearly Riemann-int.
on $[a,b]$. So we can use what we know for non-negative
functions, ~~and~~ deduce that g is Leb. int. on $[a,b]$ and
$$\int_{[a,b]} g \, d\mu = \int_a^b g(x)\,dx = \int_a^b f(x)\,dx + \underbrace{\int_a^b M\,dx}_{M(b-a)}$$

Now, if $h(x) := -M$ for $x \in [a,b]$, then h is Leb. int. on $[a,b]$:
[Indeed, $h^+ = 0$ and $h^- = M$ on $[a,b]$, and both are Leb. int.
on $[a,b]$ (since they are Riem. int. on $[a,b]$).]
Moreover,
$$\int_{[a,b]} h \, d\mu = \int_{[a,b]} h^+ \, d\mu - \int_{[a,b]} h^- \, d\mu = 0 - M \cdot \underbrace{\mu([a,b])}_{(b-a)}$$
$$= -M(b-a).$$

We ~~[illegible]~~ that $f = g + h$ is Leb. int. on $[a,b]$, and
$$\int_{[a,b]} f \, d\mu = \int_{[a,b]} g \, d\mu + \int_{[a,b]} h \, d\mu = \int_a^b f(x)\,dx + \cancel{M(b-a)} - \cancel{M(b-a)}$$

Ex. Exercise 7 Assume $f: X \to \mathbb{F}$ is integrable (w.r.t. μ) and $\alpha \in \mathbb{F}$. Then $\int \alpha f \, d\mu = \alpha \int f \, d\mu$:

Assume first $\mathbb{F} = \mathbb{R}$

- If $\alpha = 0$, then we obviously have equality.
- Assume $\alpha > 0$. Then $(\alpha f)^+ = \alpha f^+$, $(\alpha f)^- = \alpha f^-$, so

$$\begin{aligned}\int \alpha f \, d\mu &= \int (\alpha f)^+ \, d\mu - \int (\alpha f)^- \, d\mu \\ &= \int \alpha f^+ \, d\mu - \int \alpha f^- \, d\mu \\ &= \alpha \int f^+ \, d\mu - \alpha \int f^- \, d\mu = \alpha \int f \, d\mu .\end{aligned}$$

- Assume $\alpha < 0$. Then $(\alpha f)^+ = (-\alpha) f^-$, $(\alpha f)^- = (-\alpha) f^+$, so

$$\begin{aligned}\int \alpha f \, d\mu &= \int (-\alpha) f^- \, d\mu - \int (-\alpha) f^+ \, d\mu \\ &= (-\alpha) \int f^- \, d\mu - (-\alpha) \int f^+ \, d\mu \\ &= \alpha \left(\int f^+ \, d\mu - \int f^- \, d\mu \right) = \alpha \int f \, d\mu .\end{aligned}$$

Assume now $\mathbb{F} = \mathbb{C}$. Write $\alpha = a + ib$, $a, b \in \mathbb{R}$.

Then $\alpha f = (a+ib)(\mathrm{Re} f + i \, \mathrm{Im} f) = (a \, \mathrm{Re} f - b \, \mathrm{Im} f) + i (b \, \mathrm{Re} f + a \, \mathrm{Im} f)$

$$\begin{aligned}\text{So } \int \alpha f \, d\mu &= \int \mathrm{Re}(\alpha f) \, d\mu + i \int \mathrm{Im}(\alpha f) \, d\mu \\ &= \int (a \, \mathrm{Re} f - b \, \mathrm{Im} f) \, d\mu + i \int (b \, \mathrm{Re} f + a \, \mathrm{Im} f) \, d\mu \\ &= a \int \mathrm{Re} f \, d\mu - b \int \mathrm{Im} f \, d\mu + i \left(b \int \mathrm{Re} f \, d\mu + a \int \mathrm{Im} f \, d\mu \right) \\ &= (a + ib) \left(\int \mathrm{Re} f \, d\mu + i \int \mathrm{Im} f \, d\mu \right) \\ &= \alpha \int f \, d\mu .\end{aligned}$$

We can now use that the integral is linear on real-valued integrable fnct.

Extra Exercise 8

$E \in \mathcal{A}$, $\mathcal{A}_E = \{B \in \mathcal{A} \mid B \subseteq E\}$

We use results from Ex. Exercises 5 and 6

μ_E = the restr. of μ to $\mathcal{A}_E$

$\mathcal{M}^{\mathbb{C}} = \{ f : X \to \mathbb{C} \mid f \text{ is measurable (w.r.t. } \mathcal{A})\}$

$\mathcal{M}_E^{\mathbb{C}} = \{ h : E \to \mathbb{C} \mid h \text{ — '' — (w.r.t. } \mathcal{A}_E)\}$.

a) Let $f \in \mathcal{M}^{\mathbb{C}}$, $B \in \mathcal{A}_E$.

Then $\mathrm{Re}(f_E) = (\mathrm{Re} f)_E$, $\mathrm{Im}(f_E) = (\mathrm{Im} f)_E$

Since $\mathrm{Re} f$ and $\mathrm{Im} f$ are meas. (w.r.t. $\mathcal{A}$), we know that $(\mathrm{Re} f)_E$ and $(\mathrm{Im} f)_E$ are meas. (w.r.t. $\mathcal{A}_E$).

So we can conclude that f_E is meas. (w.r.t. $\mathcal{A}_E$)

Assume now f is integrable (w.r.t. μ).

Set $g = |f|$. Then g is int. (w.r.t. μ) and $|f_E| = g_E$.

So we get that $\int |f_E| \, d\mu_E = \int g_E \, d\mu_E = \int_E g \, d\mu = \int_E |f| \, d\mu \leq \int_X |f| \, d\mu < \infty$

Hence f_E is int. (w.r.t. μ_E)

Moreover,

$$\int_B f_E \, d\mu_E = \int_B \underbrace{\mathrm{Re}(f_E)}_{=(\mathrm{Re} f)_E} d\mu_E + i \int_B \underbrace{\mathrm{Im}(f_E)}_{=(\mathrm{Im} f)_E} d\mu_E$$

$$= \int_B ((\mathrm{Re} f)^+)_E \, d\mu_E - \int_B ((\mathrm{Re} f)^-)_E \, d\mu_E + i \left(\int_B ((\mathrm{Im} f)^+)_E \, d\mu_E - \int_B ((\mathrm{Im} f)^-)_E \, d\mu_E \right)$$

$$= \int_B (\mathrm{Re} f)^+ \, d\mu - \int_B (\mathrm{Re} f)^- \, d\mu + i \left(\int_B (\mathrm{Im} f)^+ \, d\mu - \int_B (\mathrm{Im} f)^- \, d\mu \right)$$

$$= \int_B \mathrm{Re} f \, d\mu + i \int_B \mathrm{Im} f \, d\mu = \int_B f \, d\mu .$$

Ex. Exercise 8

b) μ^E = the measure on $\mathcal{A}$ given by $\mu^E(A) = \mu(A \cap E)$, $A \in \mathcal{A}$.

Let $f \in \mathcal{M}^{\mathbb{C}}$. Assume f is integrable (w.r.t. μ), then

f is also integrable with respect to μ^E :

Indeed, $\int |f| \, d\mu^E = \int_E |f| \, d\mu \leq \int |f| \, d\mu < \infty$

The converse is not always true :

F. ex, let μ = the Leb. meas. on $\mathbb{R}$, $E = [0,1]$;

$f = 1_{\mathbb{R}}$. Then f_E is int. w.r.t. μ^E since

$$\int_E |f_E| \, d\mu^E = \int_{[0,1]} 1_{[0,1]} \, d\mu^E = \mu([0,1]) = 1 < \infty.$$

But f is not int. w.r.t. μ since $\int_{\mathbb{R}} |f| \, d\mu = \mu(\mathbb{R}) = \infty$.

c) Assume $f \in \mathcal{M}^{\mathbb{C}}$ is int. (w.r.t. μ^E).

Then f_E is int. (w.r.t. μ_E) since

$$\int_E |f_E| \, d\mu_E = \int_E (|f|)_E \, d\mu_E = \int_E |f| \, d\mu = \int |f| \, d\mu^E < \infty .$$

Moreover, if $A \in \mathcal{A}$, then

$$\underline{\underline{\int_A f \, d\mu^E}} = \int_A (\mathrm{Re} f)^+ \, d\mu^E - \int_A (\mathrm{Re} f)^- \, d\mu^E + i\left(\int_A (\mathrm{Im} f)^+ \, d\mu^E - \int_A (\mathrm{Im} f)^- \, d\mu^E \right)$$

$$= \int_{A\cap E} \underbrace{\left((\mathrm{Re} f)^+\right)_E}_{((\mathrm{Re} f)_E)^+} d\mu_E - \int_{A\cap E} \underbrace{\left((\mathrm{Re} f)^-\right)_E}_{((\mathrm{Re} f)_E)^-} d\mu_E + i \left(\; - - - . \; \right)$$

$$= \int_{A\cap E} (\mathrm{Re} f)_E \, d\mu_E + i \int_{A\cap E} (\mathrm{Im} f)_E \, d\mu_E$$

$$= \cdots = \underline{\underline{\int_{A\cap E} f_E \, d\mu_E}}$$

d) Along the same lines!

Ex. Exercise 9 $\quad \rho : X \to [0,\infty]$ measurable

Knows that $\nu : \mathcal{A} \to [0,\infty]$ given by $\nu(A) = \int_A \rho \, d\mu$, $A \in \mathcal{A}$ is a measure on $\mathcal{A}$.

a) Let $f \in \overline{\mathcal{M}}^+$. Then $\boxed{\int f \, d\nu = \int f \rho \, d\mu}$ $\quad$ So "$d\nu = \rho \, d\mu$"

1) This is true when $f = 1_A$, $A \in \mathcal{A}$: Indeed

$$\int 1_A \, d\nu = \nu(A) = \int_A \rho \, d\mu = \int 1_A \rho \, d\mu.$$

2) By linearity of integrals, this remains true when $f \in \mathcal{S}^+$.

3) In general, pick $\{f_n\} \subseteq \mathcal{S}^+$, $f_n \nearrow f$ pointwise on X. Using the MCT we get

$$\int f \, d\nu = \lim_{n\to\infty} \int f_n \, d\nu = \lim_{n\to\infty} \int f_n \rho \, d\mu = \int f \rho \, d\mu$$

the MCT with $f_n \rho \nearrow f \rho$

b) Let $f : X \to \mathbb{F}$ be measurable.

Then $\int |f| \, d\nu = \int |f| \rho \, d\mu = \int |f\rho| \, d\mu$, so we get that f is int. (w.r.t. ν) $\Leftrightarrow$ $f\rho$ is int. (w.r.t. μ), in which case we have for all $A \in \mathcal{A}$:

$$\int_A f \, d\nu = \int_A (\mathrm{Re} f)^+ \, d\nu - \int_A (\mathrm{Re} f)^- \, d\nu + i \left(\int_A (\mathrm{Im} f)^+ \, d\nu - \int_A (\mathrm{Im} f)^- \, d\nu \right)$$

$$= \int_A \underbrace{(\mathrm{Re} f)^+ \rho}_{\mathrm{Re}(f\rho)^+} \, d\mu - \int_A \underbrace{(\mathrm{Re} f)^- \rho}_{\mathrm{Re}(f\rho)^-} \, d\mu + i \left(\int_A \underbrace{(\mathrm{Im} f)^+ \rho}_{\mathrm{Im}(f\rho)^+} \, d\mu - \int_A \underbrace{(\mathrm{Im} f)^- \rho}_{\mathrm{Im}(f\rho)^-} \, d\mu \right)$$

$$= \int_A f \rho \, d\mu, \text{ as desired.}$$