7.6.5 $f: X \to \mathbb{R}$, $A \in \mathcal{A}$, f is int. over A (i.e. $f 1_A$ is int. w.r.t. μ)

$\{A_n\}$ seq. of disj. sets in $\mathcal{A}$.

$A = \bigcup_{n=1}^{\infty} A_n$.

Then $\int_A f \, d\mu = \sum_{n=1}^{\infty} \int_{A_n} f \, d\mu$:

Set $f_k = f \cdot 1_{\bigcup_{n=1}^{k} A_n}$ for $k \in \mathbb{N}$.

$$= \sum_{n=1}^{k} f 1_{A_n}$$

Then $f_k \to f \cdot 1_A$ (pointwise) and

$$|f_k| \leq |f| \cdot 1_A = \underbrace{|f 1_A|}_{=: g} \quad \text{for all } k \in \mathbb{N}.$$

But $\int g \, d\mu = \int |f| \cdot 1_A \, d\mu < \infty$ since f is int. over A.

So we can apply LDCT and obtain

$$\int_A f \, d\mu = \int f 1_A \, d\mu = \lim_{k \to \infty} \int f_k \, d\mu$$

$$= \lim_{k \to \infty} \int f \cdot \sum_{n=1}^{k} 1_{A_n} \, d\mu$$

$$= \lim_{k \to \infty} \sum_{n=1}^{k} \int_{A_n} f \, d\mu = \sum_{n=1}^{\infty} \int_{A_n} f \, d\mu .$$

7.6.7 "Generalized" LDCT

Assume $g \in \overline{\mathcal{M}}^+$, $\int g \, d\mu < \infty$.
$\{f_n\}_{n \in \mathbb{N}} \subseteq \overline{\mathcal{M}}$, $f \in \overline{\mathcal{M}}$, $f_n \to f$ a.e., and
$|f_n| \le g$ a.e. for all $n \in \mathbb{N}$

Then $\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$

Proof: Let $A_0 = \{x \in X \mid f_n(x) \to f(x)\}$
$A_n = \{x \in X \mid |f_n(x)| \le g(x)\}$; $n \in \mathbb{N}$.

The assumptions say that $\mu((A_k)^c) = 0$ for all $k \ge 0$.

Set $A = \bigcap_{k=0}^{\infty} A_k$. Then

$$\mu(A^c) = \mu\Big(\bigcup_{k=0}^{\infty} (A_k)^c\Big) \le \sum_{k=0}^{\infty} \underbrace{\mu((A_k)^c)}_{=0} = 0$$

So $\mu(A^c) = 0$. Hence

$$\int f \, d\mu = \int_A f \, d\mu + \underbrace{\int_{A^c} f \, d\mu}_{=0}$$

$$= \int f 1_A \, d\mu \underset{\text{LDCT}}{=} \lim_{n\to\infty} \underbrace{\int f_n 1_A \, d\mu}_{\int_A f_n \, d\mu} \underset{\text{since } \mu(A^c)=0}{=} \lim_{n\to\infty} \int f_n \, d\mu$$

Since $f_n \cdot 1_A \to f \cdot 1_A$ pt. wise
and $|f_n 1_A| \le g \cdot 1_A$ for all n and
$\int g \cdot 1_A \, d\mu \underset{\text{since } \mu(A^c)=0}{=} \int g \, d\mu < \infty$

7.6.8 Assume $g : \mathbb{R} \times X \to \mathbb{R}$ is cont. in the 1. variable and $x \to g(t,x)$ is int. (w.r.t. μ) for each $t \in \mathbb{R}$.

Assume also $\frac{\partial g}{\partial t}(t,x)$ exists for all (t,x) and that there is $h \in \bar{\mathcal{M}}^+$ s.t. $\int h \, d\mu < \infty$ and

$$\left| \frac{\partial g}{\partial t}(t,x) \right| \leq h(x) \quad \text{for all } (t,x).$$

Set $f(t) = \int g(t,x) \, d\mu(x)$ for each $t \in \mathbb{R}$.

Then $\boxed{\begin{array}{l} f \text{ is differentiable everywhere,} \\ x \to \frac{\partial g}{\partial t}(t,x) \text{ is int. (w.r.t. } \mu) \text{ for every } t \in \mathbb{R} \text{, and} \\ f'(t) = \int \frac{\partial g}{\partial t}(t,x) \, d\mu(x) \text{ for every } t \in \mathbb{R} \end{array}}$

Proof: Fix $t_0 \in \mathbb{R}$. Let $\{t_n\} \subseteq \mathbb{R}$ be s.t. $t_n \to t_0$ and $t_n \neq t_0$ for all n.

We then have that

$$\frac{\partial g}{\partial t}(t_0, x) = \lim_{n \to \infty} \underbrace{\boxed{\frac{g(t_n, x) - g(t_0, x)}{t_n - t_0}}}_{=: h_n(x)} \quad \text{for all } x \in X.$$

Each h_n is measurable, so $x \to \frac{\partial g}{\partial t}(t_0, x)$ is measurable.

Moreover, we have that $|h_n| \leq h$ for each $n \in \mathbb{N}$:

Indeed, let $n \in \mathbb{N}$ and $x \in X$. By the mean-value theorem we have that

$$\underbrace{\boxed{\frac{g(t_n, x) - g(t_0, x)}{t_n - t_0}}}_{h_n(x)} = \frac{\partial g}{\partial t}(c, x) \quad \text{for some } c \text{ between } t_0 \text{ and } t_n.$$

Thus $|h_n(x)| \leq \left| \frac{\partial g}{\partial t}(c,x) \right| \leq h(x)$, as asserted.

We can now apply LDCT and get that each h_n is integrable, $\frac{\partial g}{\partial t}(t_0, \cdot) = \lim_{n \to \infty} h_n(\cdot)$ is integrable and

$$\underbrace{\lim_{n \to \infty} \int h_n \, d\mu}_{\shortparallel} = \int \frac{\partial g}{\partial t}(t_0, x) \, d\mu(x).$$

$$\lim_{n \to \infty} \frac{\int g(t_n, x) \, d\mu(x) - \int g(t_0, x) \, d\mu(x)}{t_n - t_0}$$

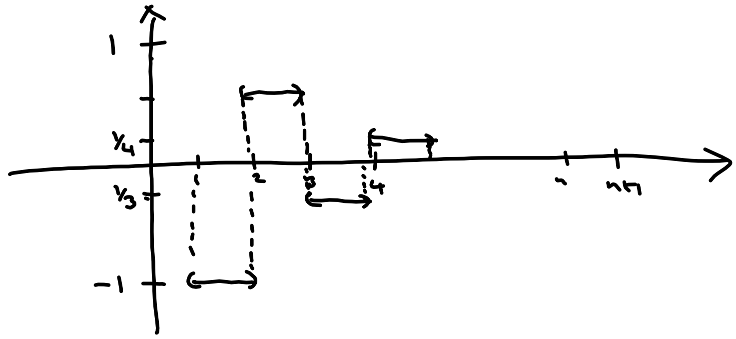
$$= \lim_{n \to \infty} \frac{f(t_n) - f(t_0)}{t_n - t_0}$$

This shows that f is differentiable at t_0, and $f'(t_0) = \int g(t_0, x) \, d\mu(x)$.

Extra Ex. 10

Let $f: \mathbb{R} \to \mathbb{R}$ be given by

$$f = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \mathbb{1}_{(n,n+1)} \quad (\text{pt.wise}), \text{ ie. } f(x) = \begin{cases} \frac{(-1)^n}{n} & \text{if } n < x < n+1 \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$



μ = Leb. measure on $\mathbb{R}$.

- Knows that f is int. (w.r.t. μ) $\iff$ $|f|$ is int. (w.r.t. μ).

Now $|f| = \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{1}_{(n,n+1)}$, so $\int |f| \, d\mu \underset{\substack{\uparrow \\ \text{Cor. of the MCT}}}{=} \sum_{n=1}^{\infty} \int \frac{1}{n} \mathbb{1}_{(n,n+1)} \, d\mu = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

Hence f is not int. (w.r.t. μ).

Let $b > 1$. Let N be the least integer which is larger or equal to b.

Then

$$\int_0^b f(x)\,dx = \int_0^{N-1} f(x)\,dx + \int_{N-1}^{b} f(x)\,dx$$

$$= \underbrace{\sum_{n=1}^{N-1} \frac{(-1)^n}{n}}_{\substack{\downarrow \\ -\ln 2 \\ \text{when } b \to \infty}} + \underbrace{\frac{(-1)^N}{N}\big(b-(N-1)\big)}_{\substack{\downarrow \\ 0 \\ \text{as } b \to \infty}} \quad \left(\text{since } \left| \frac{(-1)^N}{N}\big(b-(N-1)\big) \right| \le \frac{1}{N} \to 0 \text{ as } N \to \infty \right)$$

Note that $N \to \infty$ when $b \to \infty$

So $\int_0^{\infty} f(x)\,dx = -\ln 2$.

<u>8.1.2</u> $X = \{1,2\}$, $\mathcal{R} = \{\emptyset, \{1\}, \{1,2\}\}$

$\rho : \mathcal{R} \to \overline{\mathbb{R}}_+$ def. by $\begin{cases} \rho(\emptyset) = 0 \\ \rho(\{1\}) = 2 \\ \rho(\{1,2\}) = 1 \end{cases}$

Let μ^* be the outer measure on $\mathcal{P}(X)$ ass. with $\mathcal{R}$ and ρ.

By def. $\mu^*(\{1\}) = \inf\left\{ \sum_{j=1}^{\infty} \rho(C_j) \mid \{C_j\}_{j\in\mathbb{N}} \subseteq \mathcal{R},\ \bigcup_{j=1}^{\infty} C_j \supseteq \{1\} \right\}$

If $\{C_j\}$ is as above, then at least one of the C_j's contains 1, i.e. at least one is $\{1\}$ or $\{1,2\}$, so we get that

$$\sum_{j=1}^{\infty} \rho(C_j) \geq 1 .$$

Choosing $C_1 = \{1,2\}$, $C_j = \emptyset$ for $j \geq 2$, we get

$$\sum_{j=1}^{\infty} \rho(C_j) = \rho(C_1) + 0 + 0 + \cdots = 1 .$$

So $\underline{\mu^*(\{1\})} = 1 < 2 = \underline{\rho(\{1\})}$.

(This shows that $\mu^*|_{\mathcal{R}} \neq \rho$)

8.1.4 Assume $(X, \mathcal{R}, \rho)$ is a measure space, and let $(X, \overline{\mathcal{R}}, \overline{\rho})$ denote its completion.

Let μ^* be the outer measure on $\mathcal{P}(X)$ ass. with $\mathcal{R}$ and ρ.

Then $\mu^*(A) = \overline{\rho}(A)$ for all $A \in \overline{\mathcal{R}}$ (i.e. $\mu^*|_{\overline{\mathcal{R}}} = \overline{\rho}$)

Proof: Let $A \in \overline{\mathcal{R}}$. So $A = A' \cup N$ where $A' \in \mathcal{R}$ and $N \subseteq B$ for some $B \in \mathcal{R}$ with $\rho(B) = 0$

Let $C_1 = A'$, $C_2 = B$, $C_j = \emptyset$ for all $j \geq 3$.

Then $\{C_j\} \subseteq \mathcal{R}$ and $\bigcup_{j=1}^{\infty} C_j = A' \cup B \supseteq A$.

So $\mu^*(A) \leq \sum_{j=1}^{\infty} \rho(C_j) = \rho(A') + \underbrace{\rho(B)}_{0} = \rho(A') \overset{\text{by def. of } \overline{\rho}}{=} \overline{\rho}(A)$

i.e. $\mu^*(A) \leq \overline{\rho}(A)$

Consider now $\{C_j\} \subseteq \mathcal{R}$ s.t. $\bigcup_{j=1}^{\infty} C_j \supseteq A$. Then we have that

$$\overline{\rho}(A) \leq \overline{\rho}\left(\bigcup_{j=1}^{\infty} C_j\right) \leq \sum_{j=1}^{\infty} \overline{\rho}(C_j) = \sum_{j=1}^{\infty} \rho(C_j)$$

Taking the inf over all $\{C_j\}$ as above, we get

$\overline{\rho}(A) \leq \mu^*(A)$.

Hence $\mu^*(A) = \overline{\rho}(A)$ as desired.