Extra exercise 12

$\mathcal{S}$ semi-algebra of subsets of X, λ premeasure on $\mathcal{S}$.

Let $\mathcal{R}$ be the algebra associated with $\mathcal{S}$ and let ρ be the premeasure on $\mathcal{R}$ extending λ.

Let ν^* be the outer measure on $\mathcal{P}(X)$ ass. with $\mathcal{S}$ and λ, i.e.

$$\nu^*(A) = \inf\left\{ \sum_{j=1}^{\infty} \lambda(S_j) \;\middle|\; \{S_j\} \text{ seq. in } \mathcal{S},\ \left(\bigcup_{j=1}^{\infty} S_j\right) \supseteq A \right\}.$$

for $A \subseteq X$.

Let μ^* be the outer measure ass. with $\mathcal{R}$ and ρ, so

$$\mu^*(A) = \inf\left\{ \sum_{j=1}^{\infty} \rho(R_j) \;\middle|\; \{R_j\} \text{ seq. in } \mathcal{R},\ \left(\bigcup_{j=1}^{\infty} R_j\right) \supseteq A \right\}.$$

Then $\nu^* = \mu^*$:

Let $A \subseteq X$. Since $\mathcal{S} \subseteq \mathcal{R}$ and $\rho|_{\mathcal{S}} = \lambda$, it is clear that $\mu^*(A) \leq \nu^*(A)$.

Now, let $\{R_j\}$ be a seq. in $\mathcal{R}$ s.t. $\left(\bigcup_{j=1}^{\infty} R_j\right) \supseteq A$.

For $j \in \mathbb{N}$, we can write $R_j = \bigcup_{k=1}^{N_j} S_{j,k}$ where $S_{j,1}, \dots, S_{j,N_j} \in \mathcal{S}$ are disjoint.

Let $J = \{ (j,k) \in \mathbb{N} \times \mathbb{N} \mid j \in \mathbb{N},\ 1 \leq k \leq N_j \}$.

Then $\{S_{(j,k)}\}_{(j,k) \in J} \subseteq \mathcal{S}$, $\underbrace{\left(\bigcup_{(j,k) \in J} S_{(j,k)}\right)}_{= \bigcup_{j=1}^{\infty} R_j} \supseteq A$

So $\nu^*(A) \leq \sum_{(j,k) \in J} \lambda(S_{(j,k)}) = \sum_{j=1}^{\infty} \underbrace{\sum_{k=1}^{N_j} \lambda(S_{(j,k)})}_{\rho(R_j)}$

$$= \sum_{j=1}^{\infty} \rho(R_j)$$

Taking the inf over all $\{R_j\}$ as above, we get

$\nu^*(A) \leq \mu^*(A)$.

Altogether, $\nu^*(A) = \mu^*(A)$.

Extra Exercise 13

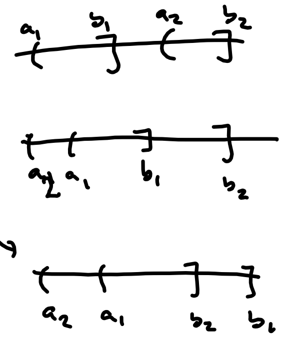
Let $\mathcal{S} = \{\emptyset\} \cup \{(a,b] : a,b \in \mathbb{R}, a<b\} \cup \{(a,\infty) \mid a \in \mathbb{R}\}$
$\cup \{(-\infty, b] \mid b \in \mathbb{R}\}$.

a) $\mathcal{S}$ is a semi-algebra:

1) $\mathcal{S}$ is closed under finite intersections: have to show that
$S_1, S_2 \in \mathcal{S} \Rightarrow S_1 \cap S_2 \in \mathcal{S}$.

- If S_1 or S_2 is empty, this is obvious.
- $S_1 = (a_1, b_1]$, $S_2 = (a_2, b_2]$.

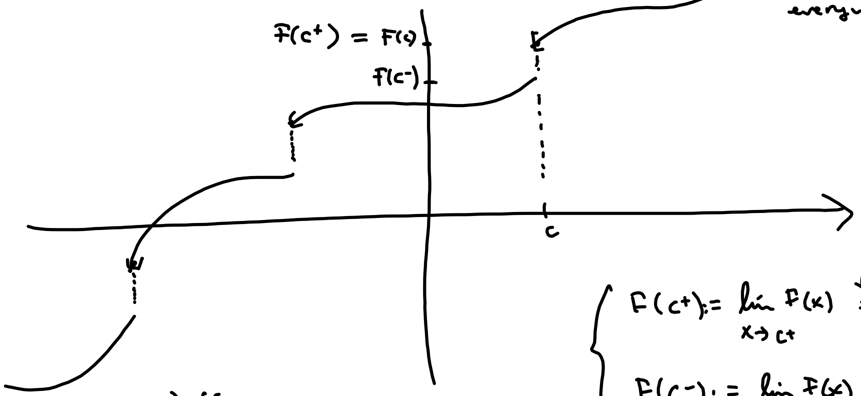
Then
$$S_1 \cap S_2 = \begin{cases} \emptyset & \text{if } b_1 \le a_2 \\ (a_1, b_1] & \text{if } a_2 \le a_1 < b_1 \le b_2 \\ (a_1, b_2] & \text{if } a_2 \le a_1 < b_2 \le b_1 \\ \vdots & \\ \vdots & \end{cases}$$



2) $S \in \mathcal{S} \Rightarrow S^c$ is a finite disj. union of sets in $\mathcal{S}$

$a<b$
$$\begin{aligned} \emptyset^c &= \mathbb{R} = (-\infty, 0] \cup (0, \infty) && \text{o.k.} \\ (a,b]^c &= (-\infty, a] \cup (b, \infty) && \text{—"—} \\ (a,\infty)^c &= (-\infty, a] && \text{—"—} \\ (-\infty, b]^c &= (b, \infty) && \text{—"—} \end{aligned}$$

b) Let from now on $F:\mathbb{R}\to\mathbb{R}$ be increasing and right-continuous everywhere.



$$\begin{cases} F(c^+) := \lim\limits_{x\to c^+} F(x) \overset{F \text{ right cont. at } c}{=} F(c) \\ F(c^-) := \lim\limits_{x\to c^-} F(x) \end{cases}$$

Define

$$\lambda_F : \mathcal{S} \to [0,\infty] \text{ by}$$

$$\begin{cases} \lambda_F(\emptyset) = 0 \\ \lambda_F((a,b]) = F(b) - F(a) \\ \lambda_F((a,\infty)) = F(\infty) - F(a) & \text{where } F(\infty) := \lim\limits_{x\to\infty} F(x) \\ \lambda_F((-\infty,b]) = F(b) - F(-\infty) & \text{where } F(-\infty) := \lim\limits_{x\to-\infty} F(x) \end{cases}$$

Then λ_F is a premeasure on $\mathcal{S}$

We use the same notation as in the constr. of the Lebesgue measure (corresponding to the case $F(x) = x$).

1) We first note that λ_F is finitely additive (on disj. unions): This is straightforward. For example, assume $S = (a,\infty)$, $S = \bigcup_{j=1}^n S_j$, $S_j \in \mathcal{S}$, disjoint. We may then assume that

$$\begin{cases} S_j = (a_{j-1}, a_j] & \text{for } j = 1,\dots,n-1 \\ S_n = (a_{n-1},\infty) & \text{where } a = a_0 < a_1 < \dots < a_{n-1}. \end{cases}$$

Then we have

$$\sum_{j=1}^{n} \lambda_F(S_j) = \underbrace{\left(\sum_{j=1}^{n-1} \big(F(a_j) - F(a_{j-1})\big)\right)}_{} + F(\infty) - \cancel{F(a_{n-1})}$$

$$\underbrace{\begin{aligned} &F(\cancel{a_1}) - F(a_0) \\ +\ &F(\cancel{a_2}) - F(\cancel{a_1}) \\ +\ &\dots \\ +\ &F(a_{n-1}) - F(\cancel{a_{n-2}}) \end{aligned}}_{\cancel{F(a_{n-1})} - F(a)}$$

$$= F(\infty) - F(a) = \lambda_F(\underbrace{(a,\infty)}_{S}).$$

2) We can now extend λ_F to $\rho_F : \mathcal{R} \to [0,\infty]$ by $\rho_F(A) := \sum_{j=1}^n \lambda_F(S_j)$ for $A = \bigcup_{j=1}^n S_j$, $S_1,\dots,S_n \in \mathcal{S}$ disjoint

Then, here also, ρ_F is well-def. and finitely add., so ρ_F is monotone and finitely subadditive.

3)

3) Consider $S \in \mathcal{S}$, $\{S_j\}_{j\in\mathbb{N}}$ seq. of disjoint sets in $\mathcal{S}$ s.t.

$$S = \bigcup_{j=1}^{\infty} S_j .$$

Have to show that $\underline{\lambda_F(S) = \sum_{j=1}^{\infty} \lambda_F(S_j)}$ $(*)$

If $S = \emptyset$, this is trivial. So assume $S \neq \emptyset$.

Consider $S = (a,b]$, $a<b$, $a,b \in \mathbb{R}$.

The proof that $\sum_{j=1}^{\infty} \lambda_F(S_j) \leq \lambda_F(S)$ goes as before.

To prove the converse inequality, let $\varepsilon > 0$.

Since F is right cont. at a, we can find $0 < \delta < b-a$ s.t. $F(a+\delta) - F(a) < \varepsilon$.

This gives that

$$\begin{aligned} \underline{\lambda_F(S)} &= \lambda_F((a,a+\delta]) + \lambda_F((a+\delta,b]) \\ &= \lambda_F((a+\delta,b]) + \underbrace{F(a+\delta)-F(a)}_{<\varepsilon} \\ &\underline{< \lambda_F((a+\delta,b]) + \varepsilon} \end{aligned}$$

Now, since F is right-cont. at each b_j, we can find $\delta_j > 0$ s.t. $\underbrace{F(b_j+\delta_j) - F(b_j)}_{=\lambda_F((b_j,b_j+\delta_j])} < \varepsilon/2^j$ for $j \in \mathbb{N}$.

Set $c_j = b_j + \delta_j$ for each j.

We then have that $\underbrace{[a+\delta,b]}_{\text{Compact}} \subseteq (a,b] \subseteq \bigcup_{j=1}^{\infty} (a_j, c_j)$

So we can find $M \in \mathbb{N}$ s.t. $[a+\delta,b] \subseteq \bigcup_{j=1}^{M} (a_j,c_j)$,

hence $\underline{(a+\delta,b] \subseteq \bigcup_{j=1}^{M} (a_j,c_j]}$

Thus we get that

$$\begin{aligned} \lambda_F((a+\delta,b]) = \rho_F((a+\delta,b]) &\leq \rho_F\Big(\bigcup_{j=1}^{M}(a_j,c_j]\Big) \\ &\leq \sum_{j=1}^{M} \underbrace{\rho_F((a_j,c_j])}_{\lambda_F((a_j,c_j])} \\ &= \sum_{j=1}^{M} \Big(\lambda_F((a_j,b_j]) + \underbrace{\lambda_F((b_j,b_j+\delta_j])}_{<\varepsilon/2^j}\Big) \\ &\leq \Big(\sum_{j=1}^{\infty} \lambda_F((a_j,b_j])\Big) + \varepsilon . \end{aligned}$$

Altogether, we get

$$\lambda_F((a,b]) \leq \Big(\sum_{j=1}^{\infty} \lambda_F((a_j,b_j])\Big) + 2\varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, we get the desired inequality.

The cases $S = (a,+\infty)$ and $S = (-\infty,b]$ are now proven as before.

From b) and Car. ext. thm for semi-algebras, we get that λ_F extends to a compl. measure μ_F on a σ-alg. $\mathcal{A}_F$ containing $\mathcal{S}$, hence also $\Sigma(\mathcal{S}) = \mathcal{B}_{\mathbb{R}}$.

μ_F is called the Leb. Stieltjes measure.

We consider only its restr. to $\mathcal{B}_{\mathbb{R}}$ in what follows.

c) μ_F is σ-finite, since $\mathbb{R} = \bigcup_{n=1}^{\infty} (-n, n]$

and $\mu_F((-n,n]) = F(n) - F(-n) < \infty$ for all $n \in \mathbb{N}$.

This implies that μ_F is the unique measure on $\mathcal{B}_{\mathbb{R}}$ which extends λ_F.

d) $$\mu_F((a,b)) = \lim_{n\to\infty} \underbrace{\mu_F\left(\left(a, b - \tfrac{(b-a)}{n+1}\right]\right)}_{F\left(b - \frac{(b-a)}{n+1}\right) - F(a)} = F(b^-) - F(a).$$

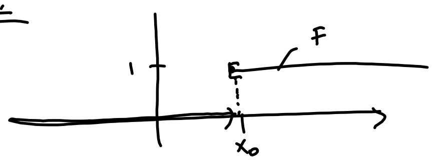
$$\begin{aligned} \mu_F(\{b\}) &= \mu_F((a,b]) - \mu_F((a,b)) \\ &= F(b) - F(a) - F(b^-) + F(a) \\ &= F(b) - F(b^-) \end{aligned}$$

This gives that

$$\begin{aligned} \mu_F(\{b\}) = 0 &\iff F(b) = F(b^-) \\ &\iff F \text{ is left-cont. at } b \\ &\iff F \text{ is cont. at } b \quad (\text{since } F \text{ is assumed to be right-cont.}) \end{aligned}$$

$$\begin{aligned} \mu_F([a,b]) &= \mu_F(\{a\}) + \mu_F((a,b]) \\ &= F(a) - F(a^-) + F(b) - F(a) \\ &= F(b) - F(a^-) \end{aligned}$$

f) $F = 1_{[x_0, \infty)}$



Let $a<b$. Then

$$\begin{aligned} \mu_F((a,b]) &= 1_{[x_0,\infty)}(b) - 1_{[x_0,\infty)}(a) \\ &= \begin{cases} 0-0 & \text{if } b < x_0 \\ 1-0 & \text{if } a < x_0 \le b \\ 1-1 & \text{if } x_0 \le a \end{cases} \\ &= \begin{cases} 1 & \text{if } a < x_0 \le b \\ 0 & \text{otherwise} \end{cases} \\ &= \nu_{x_0}((a,b]) \end{aligned}$$

where ν_{x_0} is the Dirac measure at x_0.

Similarly, we get that

$\mu_F((a,\infty)) = \nu_{x_0}((a,\infty))$ for all $a, b \in \mathbb{R}$.

$\mu_F((-\infty,b]) = \nu_{x_0}((-\infty,b])$

In part. ν_{x_0} extends λ_F. By uniqueness, we get $\mu_F = \nu_F$ on $\mathcal{B}_{\mathbb{R}}$

e) Assume $F: \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$, with $g := F' \geq 0$ on $\mathbb{R}$, g continuous

This implies F is continuous on $\mathbb{R}$ and increasing.

Set $\nu(A) := \int_A g \, d\mu$, $A \in \mathcal{B}_{\mathbb{R}}$ (where μ is the Leb. measure on $\mathbb{R}$)

Then $\nu = \mu_F$:

~~Note~~ Note that

$a < b$

$$\underbrace{F(b) = F(a)}_{\text{Fund. thm of Analysis}} + \int_a^b \underbrace{F'(x)}_{g(x)} dx \underbrace{=}_{\text{since } g \text{ is Riem. int. on } [a,b]} F(a) + \int_{[a,b]} g \, d\mu$$

$$= \underbrace{\int_{\{a\}} g \, d\mu}_{\substack{= 0 \\ \text{since } \mu(\{a\}) = 0}} + \int_{(a,b]} g \, d\mu = \int_{(a,b]} g \, d\mu$$

So
$$\underbrace{\int_{(a,b]} g \, d\mu}_{\nu((a,b])} = \underbrace{F(b) - F(a)}_{\mu_F((a,b])}$$

Moreover we get then for all $a, b \in \mathbb{R}$:

$$\nu((a,\infty)) = \lim_{n\to\infty} \nu((a,n]) = \lim_{n\to\infty} \mu_F((a,n]) = \mu_F((a,\infty))$$

$(a,\infty) = \bigcup_{n=1}^{\infty} (a,n]$

Similarly, we get $\nu((-\infty,b]) = \mu_F((-\infty,b])$.

This implies that $\nu = \mu_F = \lambda_F$ on $\mathcal{J}$.

By uniqueness of μ_F, we get $\nu = \mu_F$ on $\mathcal{B}_{\mathbb{R}}$.