MAT3400/4400 - Spring 19 - Exercises for Monday, Mars 4

Extra exercise 12

Let $\mathcal{S}$ be a semi-algebra of subsets of a non-empty set X and let λ be a premeasure on $\mathcal{S}$. Let $\mathcal{R}$ be the algebra consisting of all unions of finitely many disjoint sets in $\mathcal{S}$ and let ρ be the premeasure on $\mathcal{R}$ given by $\rho(A) = \sum_{j=1}^{n} \lambda(S_j)$ whenever $A = S_1 \cup \cdots \cup S_n$ for some disjoint sets $S_1, \ldots, S_n$ in $\mathcal{S}$.

Let ν^* denote the outer measure on $\mathcal{P}(X)$ associated with $\mathcal{S}$ and λ, and let μ^* denote the outer measure on $\mathcal{P}(X)$ associated with $\mathcal{R}$ and ρ. Check that $\nu^* = \mu^*$.
(This is the content of the Remark on p. 302 in [L]).

Extra exercise 13

Let $\mathcal{S}$ denote the collection of subsets of $\mathbb{R}$ given by

$$\mathcal{S} = \big\{\emptyset\big\} \cup \big\{(a,b] \mid a,b \in \mathbb{R}, a < b\big\} \cup \big\{(-\infty, b] \mid b \in \mathbb{R}\big\} \cup \big\{(a,\infty) \mid a \in \mathbb{R}\big\}.$$

a) Check that $\mathcal{S}$ is a semi-algebra.

Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing function (i.e., we have $F(x) \leq F(y)$ whenever $x < y$) and assume that F is *right-continuous* on $\mathbb{R}$, that is, we have $F(c^+) = F(c)$ for all $c \in \mathbb{R}$, where $F(c^+) := \lim_{x\to c^+} F(x)$.

Note that if F is not continuous at $c \in \mathbb{R}$, then F is not left-continuous at c, that is, we have $F(c^-) < F(c)$, where $F(c^-) := \lim_{x\to c^-} F(x)$; hence, in this case, the graph of F makes a jump of height $F(c) - F(c^-)$ at c. (Make a drawing illustrating this!).

Set $F(\infty) := \lim_{x\to\infty} F(x) = \sup\big\{F(x) : x \in \mathbb{R}\big\} \in \mathbb{R} \cup \{\infty\}$ and
$F(-\infty) := \lim_{x\to-\infty} F(x) = \inf\big\{F(x) : x \in \mathbb{R}\big\} \in \mathbb{R} \cup \{-\infty\}$.

Define $\lambda_F : \mathcal{S} \to [0,\infty]$ by

- $\lambda_F(\emptyset) = 0\,;$
- $\lambda_F((a,b]) = F(b) - F(a)$ when $a, b \in \mathbb{R}$, $a < b\,;$
- $\lambda_F((a,\infty)) = F(\infty) - F(a)$ when $a \in \mathbb{R}\,;$
- $\lambda_F((-\infty,b]) = F(b) - F(-\infty)$ when $b \in \mathbb{R}\,.$

(Note that if $F(x) = x$ for all $x \in \mathbb{R}$, then λ_F is the usual length function on $\mathcal{S}$).

b) Show that λ_F is a premeasure on $\mathcal{S}$.

Hint: Adapt carefully the proof for the case $F(x) = x$.

Applying Carathéodory's extension theorem for semi-algebras with $\mathcal{S}$ and λ_F, we obtain that λ_F extends to a complete measure μ_F on a σ-algebra $\mathcal{A}_F$ which contains $\mathcal{S}$, and therefore also $\Sigma(\mathcal{S}) = \mathcal{B}_{\mathbb{R}}$ (the Borel subsets of $\mathbb{R}$), cf. Extra Exercise 2.

The measure μ_F is called the *Lebesgue-Stieltjes* measure associated with F. We also denote by μ_F its restriction to $\mathcal{B}_R$.

c) Check that $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu_F)$ is σ-finite. (It follows that μ_F is the unique extension of λ_F to a measure on $\mathcal{B}_{\mathbb{R}}$).

d) One can compute a formula for $\mu_F(I)$ for any kind of interval $I \subseteq \mathbb{R}$. Do this for $I = (a, b)$ and $I = [a, b]$, where $-\infty < a < b < \infty$. Give also a formula for $\mu_F(\{c\})$ for any $c \in \mathbb{R}$ and deduce that F is continuous at c if and only if $\mu_F(\{c\}) = 0$.

e) Assume that $F : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$ and that its derivative $g := F'$ is continuous and nonnegative on $\mathbb{R}$. As is well-known, F is then continuous and increasing. Let μ denote the Lebesgue measure on $\mathcal{B}_{\mathbb{R}}$. Show that

$$\mu_F(A) = \int_A g \, d\mu \quad \text{for all } A \in \mathcal{B}_{\mathbb{R}}.$$

Note that it follows now from Extra Exercise 9 that if $f : \mathbb{R} \to \mathbb{C}$ is Borel measurable, then f is integrable w.r.t. μ_F if and only if fg is integrable w.r.t. μ, in which case we have

$$\int_A f \, d\mu_F = \int_A fg \, d\mu$$

for all $A \in \mathcal{B}_{\mathbb{R}}$.

f) Let $x_0 \in \mathbb{R}$ and set $F = 1_{[x_0,\infty)}$. Note that F is increasing and right-continuous on $\mathbb{R}$. Show that μ_F is the Dirac measure at x_0 on $\mathcal{B}_{\mathbb{R}}$, that is, for $A \in \mathcal{B}_{\mathbb{R}}$, we have $\mu_F(A) = 1$ if $x_0 \in A$, and $\mu_F(A) = 0$ otherwise.