Lindstrøm 8.4.

2 a) First, note that if $I=(c,d]$ with $c,d\in\mathbb{R}$, $c<d$, then $I+a=(c+a,d+a]$, and so

$$\lambda(I+a) = d+a-(c+a) = d-c = \lambda(I).$$

The same holds if the interval has one or two infinite endpoints.

Now let $E\subseteq\mathbb{R}$ and $a\in\mathbb{R}$. Let $(I_n)_{n=1}^{\infty}$ be a sequence of half-open intervals with $\bigcup_{n=1}^{\infty} I_n \supseteq E$. Then

$$E+a \subseteq \left(\bigcup_{n=1}^{\infty} I_n\right)+a = \bigcup_{n=1}^{\infty}(I_n+a).$$

Since $(I_n+a)_{n=1}^{\infty}$ is a sequence of half-open intervals covering $E+a$, we have that $\mu^*(E) \leq \sum_{n=1}^{\infty}\lambda(I_n+a) = \sum_{n=1}^{\infty}\lambda(I_n)$.
Since $(I_n)_n$ was arbitrary, this shows that

$$\mu^*(E+a) \leq \mu^*(E).$$

Now use the inequality above with $E+a$ instead of E and $-a$ instead of a to get

$$\mu^*(E) = \mu^*((E+a)-a) \leq \mu^*(E+a).$$

Thus,

$$\mu^*(E+a) = \mu^*(E)$$

for all $E\subseteq\mathbb{R}$ and $a\in\mathbb{R}$.

b) Suppose E is measurable. Let $A\subseteq\mathbb{R}$ and $a\in\mathbb{R}$. Then

$$\begin{aligned}
&\mu^*(A\cap(E+a)) + \mu^*(A\cap(E+a)^c)\\
&= \mu^*((A-a+a)\cap(E+a)) + \mu^*((A-a+a)\cap(E^c+a))\\
\overset{a)}&{=} \mu^*((A-a)\cap E) + \mu^*((A-a)\cap E^c)\\
&= \mu^*(A-a) = \mu^*(A).
\end{aligned}$$

Since E is measurable (first equality in the last line); by a) (second equality in the last line).

This shows that $E+a$ is measurable as well.

c) By b), we have that $E+a$ is measurable when E is measurable. By a) and by definition of μ, we then have that $\mu(E+a) = \mu^*(E+a)$
$$= \mu^*(E) = \mu(E).$$

5 We begin by proving that $\mu^*(rA) = r\mu^*(A)$ for all $r>0$ and $A \subseteq \mathbb{R}$. First, note that if I is a half-open interval, say $I=(c,d]$, then $rI = (rc, rd]$, and so
$$\lambda(rI) = rd - rc = r(d-c) = r\lambda(I).$$

Now let $(I_n)_{n=1}^{\infty}$ be any sequence of half-open intervals with $\bigcup_{n=1}^{\infty} I_n \supseteq A$. Then $rA \subseteq r\left(\bigcup_{n=1}^{\infty} I_n\right) = \bigcup_{n=1}^{\infty} rI_n$. Since $(rI_n)_{n=1}^{\infty}$ is a sequence of intervals covering rA, we have
$$\mu^*(rA) \leq \sum_{n=1}^{\infty} \lambda(rI_n) = \sum_{n=1}^{\infty} r\lambda(I_n) = r\sum_{n=1}^{\infty} \lambda(I_n).$$

Since $(I_n)_{n=1}^{\infty}$ was arbitrary, we must have
$$\mu^*(rA) \leq \inf\left\{ r\sum_{n=1}^{\infty}\lambda(I_n) : (I_n)_{n=1}^{\infty} \text{ covers } A \right\}$$
$$= r \cdot \inf\left\{ \sum_{n=1}^{\infty}\lambda(I_n) : (I_n)_{n=1}^{\infty} \text{ cover } A \right\}$$
$$= r \cdot \mu^*(A). \qquad (*)$$

To obtain the reverse direction, we use $(*)$ with $1/r$ instead of r:
$$\mu^*(A) = \mu^*\left(\tfrac{1}{r} \cdot rA\right) \leq \tfrac{1}{r} \cdot \mu^*(rA).$$

This gives $r \cdot \mu^*(A) \leq \mu^*(rA)$, which finishes this part of the proof.

Now suppose A is measurable, and let $B \subseteq \mathbb{R}$. Using the identity $r(C \cap D) = rC \cap rD$, we obtain
$$\mu^*(B \cap rA) + \mu^*(B \cap (rA)^c) = \mu^*\left(r\left(\tfrac{1}{r}B \cap A\right)\right) + \mu^*\left(r\left(\tfrac{1}{r}B \cap A^c\right)\right)$$
$$= r\left[\mu^*\left(\tfrac{1}{r}B \cap A\right) + \mu^*\left(\tfrac{1}{r}B \cap A^c\right)\right] \underset{\text{A is measurable}}{=} r\mu^*\left(\tfrac{1}{r}B\right) \underset{\text{First part.}}{=} \mu^*(B)$$

This shows that rA is measurable, and by the first part we obtain
$$\mu(rA) = r\mu(A).$$

Extra Exercise 14

We consider the collections $\mathcal{B}+a$ and $r\mathcal{B}$.

First, let U be any open set. Then $U = \bigcup_{n=1}^{\infty} I_n$ where each I_n is an open interval. Now

$$U = U - a + a = \bigcup_{n=1}^{\infty} (I_n - a) + a.$$

As remarked earlier, each $I_n - a$ is again an interval, in this case open, and so $\bigcup_{n=1}^{\infty}(I_n - a)$ is an open set, hence Borel-measurable. Thus we have written U in the form

$$U = B + a; \quad B \text{ Borel-measurable}$$

and so $U \in \mathcal{B}+a$. This shows that all open sets are in $\mathcal{B}+a$.

Similarly,

$$U = r \cdot \frac{1}{r} \cdot U = r \cdot \bigcup_{n=1}^{\infty} \left(\tfrac{1}{r} I_n\right).$$

Here, $\frac{1}{r} I_n$ is also an open interval (namely $r(a,b) = (ra, rb)$ if $r>0$ and $r(a,b) = (rb, ra)$ if $r<0$), and so $U \in r\mathcal{B}$.

We now show that $\mathcal{B}+a$ and $r\mathcal{B}$ are σ-algebras. Let $(B_n + a)_{n=1}^{\infty}$ be a sequence in $\mathcal{B}+a$, where each $B_n \in \mathcal{B}$. Then

$$\bigcup_{n=1}^{\infty} (B_n + a) = \underbrace{\left(\bigcup_{n=1}^{\infty} B_n\right)}_{\text{in } \mathcal{B}} + a \quad \in \mathcal{B}+a$$

Moreover, if $B + a \in \mathcal{B}+a$ $(B \in \mathcal{B})$ then $(B+a)^c = B^c + a \in \mathcal{B}+a$ since $B^c \in \mathcal{B}$ and $\emptyset = \emptyset + a \in \mathcal{B}+a$, so $\mathcal{B}+a$ is a σ-algebra.

Similarly, if $(rB_n)_{n=1}^{\infty}$ is a sequence in $r\mathcal{B}$ ($B_n \in \mathcal{B}$ for each n) then

$$\bigcup_{n=1}^{\infty} rB_n = r \underbrace{\bigcup_{n=1}^{\infty} B_n}_{\in \mathcal{B}} \in r\mathcal{B}$$

and $(rB)^c = rB^c$ for $B \in \mathcal{B}$, so $(rB)^c \in r\mathcal{B}$ and $\emptyset = r\emptyset \in r\mathcal{B}$.

Thus, both $\mathcal{B}+a$ and $r\mathcal{B}$ are σ-algebras that contain all the open sets.

By definition of $\mathcal{B}$, we then have that

$$\mathcal{B} \subseteq \mathcal{B} + a \quad \text{and} \quad \mathcal{B} \subseteq r\mathcal{B}.$$

[We use "$a = -a$"]

It is now easy to show the reverse inclusions:
If $B \in \mathcal{B}$ then $B = B' - a$ for some $B' \in \mathcal{B}$ by the above,
and so $B + a = B' \in \mathcal{B}$, showing that $\mathcal{B} + a \subseteq \mathcal{B}$.
Similarly, if $B \in \mathcal{B}$ then $B = r^{-1}B''$ for some $B'' \in \mathcal{B}$, and so
$rB = B'' \in \mathcal{B}$, showing that $r\mathcal{B} \subseteq \mathcal{B}$. Hence we have

$$\mathcal{B} = \mathcal{B} + a = r\mathcal{B}$$

which gives the desired conclusion.

Extra Exercise 15

a) Note that for $a \in \mathbb{R}$, $b \in \mathbb{R}$:

$$\begin{aligned} f_a^{-1}([-\infty, b)) &= \{x \in \mathbb{R} : f_a(x) \in [-\infty, b)\} \\ &= \{x \in \mathbb{R} : f(x+a) \in [-\infty, b)\} \\ &= \{x \in \mathbb{R} : x + a \in f^{-1}([-\infty, b))\} \\ &= f^{-1}([-\infty, b)) - a. \end{aligned}$$

By 8.4.2, $f^{-1}([-\infty, b)) - a$ is measurable if $f^{-1}([-\infty, b))$ is,
and conversely, if $f^{-1}([-\infty, b)) - a$ is measurable, then
$f^{-1}([-\infty, a)) = (f^{-1}([-\infty, a)) - a) + a$ is measurable, again by 8.4.2.
Hence, f is (Lebesgue) measurable if and only if f_a is.
[This is the proof for $S = \overline{\mathbb{R}}$. If $S = \mathbb{C}$, take an open set $U \subseteq \mathbb{C}$
and perform the same argument].

b) Begin by letting $f = \mathbb{1}_A$ for some $A \in \mathcal{M}$.
Then $f_a = \mathbb{1}_{A-a}$, and so

$$\int_{E-a} f_a \, d\mu = \mu((E-a) \cap (A-a)) = \mu(E \cap A) = \int_E f \, d\mu.$$

Next, let $f = \sum_{k=1}^{n} c_k \mathbb{1}_{A_k}$ be a simple function, with $A_k \in \mathcal{M}$, $c_k \in \overline{\mathbb{R}}^+$, $c_k \geq 0$ for each k. Then

$$\int_{E-a} f_a \, d\mu = \int_{E-a} \sum_{k=1}^{n} c_k \mathbb{1}_{A_k - a} \, d\mu = \sum_{k=1}^{n} c_k \int_{E-a} \mathbb{1}_{A_k - a} \, d\mu$$

$$= \sum_{k=1}^{n} c_k \int_E \mathbb{1}_{A_k} \, d\mu = \int_E f \, d\mu.$$

We have now proved the proposition for simple functions, and move on to proving it for general measurable functions.
Pick an increasing sequence $(h_n)_n$ of simple functions converging pointwise to f as in Proposition 7.5.3 in Lindstrøm.
Then $((h_n)_a)_n$ is an increasing sequence of simple functions converging pointwise to f_a. By the MCT (7.5.6. in Lindstrøm),

$$\int_{E-a} f_a \, d\mu = \lim_{n\to\infty} \int_{E-a} (h_n)_a \, d\mu \underset{\substack{\uparrow \\ \text{Already proved for simple functions}}}{=} \lim_{n\to\infty} \int_E h_n \, d\mu \underset{\substack{\uparrow \\ \text{MCT again}}}{=} \int_E f \, d\mu.$$

c) If S is either $\overline{\mathbb{R}}$ or $\mathbb{C}$, then $\int_{E-a} |f_a| \, d\mu = \int_E |f| \, d\mu$ by b), and so integrability of f_a is equivalent to integrability of f.
Suppose $S = \overline{\mathbb{R}}$. Then f_+ and f_- take values in $\overline{\mathbb{R}^+}$, and so
$\int_{E-a} (f_+)_a \, d\mu = \int_E f_+ \, d\mu$ and $\int_{E-a} (f_-)_a \, d\mu = \int_E f_- \, d\mu$, both by b).
But then

$$\int_{E-a} (f)_a \, d\mu = \int_{E-a} (f_+)_a \, d\mu - \int_{E-a} (f_-)_a \, d\mu = \int_E f_+ \, d\mu - \int_E f_- \, d\mu = \int_E f \, d\mu.$$

~~If $f: \mathbb{R} \to \mathbb{C}$, write $f = Re(f) + i Im(f)$. Then by b) we get~~
~~$\int_{E-a} f_a \, d\mu = \int_{E-a} Re(f_a) \, d\mu + i \int_{E-a} Im(f_a) \, d\mu = \int_E f \, d\mu$~~

If $f: \mathbb{R} \to \mathbb{C}$, write $f = Re(f) + i\,Im(f)$. Since $Re(f), Im(f) : \mathbb{R} \to \mathbb{R}$, we get from what we already proved that

$$\int_{E-a} f_a \, d\mu = \int_{E-a} Re(f_a) \, d\mu + i \int_{E-a} Im(f_a) \, d\mu = \int_E Re(f) \, d\mu + i \int_E Im(f) \, d\mu$$

$$= \int_E f \, d\mu .$$

~~MAT3400/4400 Spring 19~~ Mars 11

~~Exercise Solutions~~

From Lindstrøm, section ~~8.4.8~~ 8.5

2 a) By Definition 7.1.1 (iii) and Proposition 7.1.2 b), the intersection and union of a countable number of measurable sets are again measurable sets, and so both G_δ and F_σ sets are measurable since open and closed sets are measurable.

b) By Proposition 8.5.1, we can for each $k \in \mathbb{N}$ find an open set $U_k \supseteq A$ such that $\mu(U_k \setminus A) < 1/k$. Now define $G = \bigcap_{k \in \mathbb{N}} U_k$. This is a G_δ-set, and

$$G \setminus A = \left(\bigcap_{k \in \mathbb{N}} U_k\right) \setminus A = \bigcap_{k \in \mathbb{N}} U_k \setminus A \subseteq U_{k_0} \setminus A$$

for every fixed $k_0 \in \mathbb{N}$. Thus, for all $k_0 \in \mathbb{N}$, we have that

$$\mu(G \setminus A) \leq \mu(U_{k_0} \setminus A) < \frac{1}{k_0}.$$

Since k_0 was arbitrary, this shows that $\mu(G \setminus A) = 0$.

c) By Proposition 8.5.1, we can for each $k \in \mathbb{N}$ find a closed set $C_k \subseteq A$ such that $\mu(A \setminus C_k) < 1/k$. Define $F = \bigcup_{k \in \mathbb{N}} C_k$. This is an F_σ-set, and

$$A \setminus F = A \setminus \left(\bigcup_{k \in \mathbb{N}} C_k\right) = A \cap \left(\bigcup_{k \in \mathbb{N}} C_k\right)^c = A \cap \left(\bigcap_{k \in \mathbb{N}} C_k^c\right)$$

$$= \bigcap_{k \in \mathbb{N}} A \cap C_k^c = \bigcap_{k \in \mathbb{N}} A \setminus C_k \subseteq A \setminus C_{k_0}$$

for every fixed $k_0 \in \mathbb{N}$. Thus, for all $k_0 \in \mathbb{N}$, we have that

$$\mu(A \setminus F) \leq \mu(A \setminus C_{k_0}) < \frac{1}{k_0}.$$

Since k_0 was arbitrary, this shows that $\mu(A \setminus F) = 0$.

Extra Exercise 16

Suppose first that $\mu(A) < \infty$.

Let $\varepsilon > 0$. By 8.5.1 and 8.5.2. in Lindstrøm, we can find an open set $G \supseteq A$ and a compact set $K \subseteq A$ with $\mu(A \setminus K) < \varepsilon$ and $\mu(G \setminus A) < \varepsilon$. Since $\mu(A) < \infty$, both of these imply that $\mu(K) < \infty$ and $\mu(G) < \infty$, and so

$$\mu(A \setminus K) = \mu(A) - \mu(K) < \varepsilon$$
$$\mu(G \setminus A) = \mu(G) - \mu(A) < \varepsilon.$$

Thus $\mu(A) = \inf\{\mu(G) : G \text{ open}, G \supseteq A\}$
$= \sup\{\mu(K) : K \text{ compact}, K \subseteq A\}$

since $\varepsilon > 0$ was arbitrary.

Now let A be a general Lebesgue measurable set.

Let $\mu(A) = \infty$. Then if $U \supseteq A$ is open, $\mu(U) = \infty$ as well, and so $\inf\{\mu(G) : G \supseteq A \text{ open}\} = \infty$.

We can write $A = \bigcup_{k=1}^{\infty} A_k$ where $A_k \cap A_\ell = \emptyset$ for $k \neq \ell$ and each A_k is measurable with $\mu(A_k) < \infty$. By 8.5.2 in Lindstrøm pick compact sets $K_k \subseteq A_k$ with $\mu(A_k \setminus K_k) < 2^{-k}$.

Then

$$\tfrac{1}{2} > \sum_{k=1}^{\infty} \mu(A_k \setminus K_k) = \sum_{k=1}^{\infty} (\mu(A_k) - \mu(K_k)).$$

Since $\infty = \mu(A) = \sum_{k=1}^{\infty} \mu(A_k)$, we must have $\sum_{k=1}^{\infty} \mu(K_k) = \infty$ as well.

Let $\tilde{K}_k = \bigcup_{j=1}^{k} K_j$. Then each $\tilde{K}_k$ is compact, $\tilde{K}_k \subseteq A$ and $\lim_{n \to \infty} \mu(\tilde{K}_k) = \infty$. Thus $\sup\{\mu(K) : K \subseteq A \text{ compact}\} = \infty$.

Extra Exercise 17

a) Suppose first that $A \subseteq \mathbb{R}$ is Lebesgue measurable. By Lindstrøm exercise 8.4.2. c), we can find a Borel measurable set $F \subseteq A$ with $m(A \setminus F) = 0$.

Since F is Borel measurable, it is Lebesgue measurable, hence $A \setminus F$ is Lebesgue measurable. We now have

$$A = F \cup (A \setminus F).$$

Let $B = A \setminus F$.
By Lindstrøm exercise 8.4.2. b), there is a Borel measurable set $G \supseteq B$ with $m(G \setminus B) = 0$. Since $m(B) = 0$, we have $\overset{\mu(a)}{m(G)} = 0$ as well. This shows that $B = A \setminus F$ is in $\mathcal{N}_\mu$. Since F is a Borel set, this finishes this direction of the proof.

Suppose next that $A = B \cup N$ where B is a Borel set and N is a null set (in $\mathcal{N}_\mu$). By Lindstrøm Theorem 8.4.5., Lebesgue measure is complete, and so ~~m~~ N is Lebesgue measurable (with $m(N) = 0$). But since both B and N are Lebesgue measurable, $A = B \cup N$ is.

b) By a), $\mathcal{L}$ is exactly equal to
$$\overline{\mathcal{B}} = \{ B \cup N : B \in \mathcal{B},\ N \in \mathcal{N}_\mu \}.$$

Furthermore, $m|_{\mathcal{B}} = \mu$. This shows that $(\mathbb{R}, \mathcal{L}, m)$ is the completion of $(\mathbb{R}, \mathcal{B}, \mu)$.

And if $A = B \cup N$ with $B \in \mathcal{B}$ and $N \in \mathcal{N}_\mu$,

then
$$m(A) = m(B) + m(N) - m(B \cap N) = m(B) = \mu(B).$$