MAT3400/4400 - Spring 19 - Exercises for Monday, Mars 11

- From Lindstrøm's book, section 8.4: 2, 5

Extra exercise 14

Let $E \subseteq \mathbb{R}$ be Borel measurable, and let $a, r \in \mathbb{R}$, $r \neq 0$. Show that $E + a$ and rE are also Borel measurable.

Hint: Consider the collections of subsets of $\mathbb{R}$ given by
$\mathcal{B} + a := \{E + a \mid E \text{ Borel subset of } \mathbb{R}\}$ and $r\mathcal{B} := \{rE \mid E \text{ Borel subset of} \mathbb{R}\}$.

Extra exercise 15

Let S denote either $\overline{\mathbb{R}}$ or $\mathbb{C}$, and let $f : \mathbb{R} \to S$. Let $a \in \mathbb{R}$ and define $f_a : \mathbb{R} \to S$ by

$$f_a(x) := f(x + a) \quad \text{for all } x \in \mathbb{R}.$$

Moreover, let E be a Lebesgue measurable subset of $\mathbb{R}$, and set

$$E - a := E + (-a) = \{e - a \mid e \in E\}.$$

Note that $E - a$ is Lebesgue measurable, as follows from [L; Exercise 8.4.2].
Let μ denote the Lebesgue measure on the Lebesgue measurable subsets of $\mathbb{R}$.

a) Show that f is Lebesgue measurable if and only if f_a is Lebesgue measurable.

b) Assume that $f : \mathbb{R} \to \overline{\mathbb{R}}_+$ is Lebesgue measurable, so $f_a : \mathbb{R} \to \overline{\mathbb{R}}_+$ is also Lebesgue measurable (by a)). Show that $\int_{E-a} f_a \, d\mu = \int_E f \, d\mu$.

c) Let again $f : \mathbb{R} \to S$. Show that f is integrable over E if and only if f_a is integrable over $E - a$, in which case we have

$$\int_{E-a} f_a \, d\mu = \int_E f \, d\mu \,,$$

in other words, we have $\int_B f(x + a) \, d\mu(x) = \int_{B+a} f(t) \, d\mu(t)$ with $B := E - a$.

- From Lindstrøm's book, section 8.5: 2

Extra exercise 16

Assume that $A \subseteq \mathbb{R}$ is Lebesgue measurable and let μ denote the Lebesgue measure on the Lebesgue measurable subsets of $\mathbb{R}$. Show that

$$\begin{aligned} \mu(A) &= \inf \big\{\mu(G) \mid G \text{ is open in } \mathbb{R} \text{ and } A \subseteq G\big\} \\ &= \sup \big\{\mu(K) \mid K \text{ is compact in } \mathbb{R} \text{ and } K \subseteq A\big\}. \end{aligned}$$

Extra exercise 17

In this exercise (and only there) we let $\mathcal{B}$ denote the σ-algebra of all Borel subsets of $\mathbb{R}$, while $\mathcal{L}$ denotes the σ-algebra of all Lebesgue measurable subsets of $\mathbb{R}$. Moreover, we let μ denote the Lebesgue measure on $\mathcal{B}$, while m denotes the Lebesgue measure on $\mathcal{L}$.
Let $\mathcal{N}_\mu$ denote the collection of all null sets for μ, that is, $N \in \mathcal{N}_\mu$ if and only if $N \subseteq \mathbb{R}$ and there exists some $C \in \mathcal{B}$ such that $N \subseteq C$ and $\mu(C) = 0$.

a) Show that $A \subseteq \mathbb{R}$ is Lebesgue measurable if and only if there exists a Borel set B and $N \in \mathcal{N}_\mu$ such that $A = B \cup N$.

b) Show that $(\mathbb{R}, \mathcal{L}, m)$ is the completion of $(\mathbb{R}, \mathcal{B}, \mu)$.