MAT3400/4400 Spring 2019 - Exercises March 18 SOLUTIONS

Notes from ELA:

2.2 Define $f_n = f \cdot \mathbb{1}_{[1,n]}$ for $n = 1, 2, \ldots$. Then for $1 \leq p < \infty$, $(|f_n|^p)_{n=1}^{\infty}$ is a sequence of pointwise increasing measurable functions that converges pointwise to $|f|^p$. By the MCT:

$$\int_{[1,\infty)} |f|^p \, d\mu = \lim_{n\to\infty} \int_{[1,\infty)} |f_n|^p \, d\mu = \lim_{n\to\infty} \int_{[1,n]} \frac{1}{x^p} \, d\mu(x).$$

By Theorem 7.5.9 in Lindstrøm, since $x \mapsto 1/x^p$ is Riemann-integrable on $[1,n]$, the Riemann integral coincides with the Lebesgue integrals, and so

$$\int_{[1,n]} \frac{1}{x^p} \, d\mu(x) = \int_1^n \frac{1}{x^p} \, dx = \begin{cases} \ln(n) & ; \quad p = 1 \\ \frac{1}{-p+1}\left(n^{-p+1} - 1\right) & ; \quad p > 1 \end{cases}$$

It follows that

$$\int_{[1,\infty)} |f|^p \, d\mu = \begin{cases} \infty & ; \quad p = 1 \\ \frac{-1}{1-p} & ; \quad p > 1 \end{cases}$$

Hence $f \in \mathcal{L}^p$ iff $p > 1$, and in that case

$$\|f\|_p = \left(\int_{[1,\infty)} |f|^p \, d\mu \right)^{\frac{1}{p}} = \frac{1}{(p-1)^{1/p}}.$$

2.3 Define $f_n = f \cdot \mathbb{1}_{[-n,n]}$ for $n=1,2,\dots$ Then $(|f_n|^p)_{n=1}^{\infty}$ is a pointwise increasing sequence of measurable functions converging pointwise to $|f|^p$ $(1 \le p < \infty)$.
By the MCT;

$$\int_{\mathbb{R}} |f|^p d\mu = \lim_{n\to\infty} \int_{\mathbb{R}} |f_n|^p d\mu = \lim_{n\to\infty} \int_{[-n,n]} |e^{-x^2}|^p d\mu(x)$$

$$= \lim_{n\to\infty} \int_{[-n,n]} e^{-px^2} d\mu(x).$$

Since $x \mapsto e^{-px^2}$ is continuous on $[-n,n]$, it is Riemann integrable there, and so the Riemann and Lebesgue integrals coincide. We get

$$\int_{[-n,n]} e^{-px^2} d\mu(x) = \int_{-n}^{n} e^{-px^2} dx \underset{\underset{u=\sqrt{p}x}{\uparrow}}{=} \frac{1}{\sqrt{p}} \int_{-\sqrt{p}n}^{\sqrt{p}n} e^{-u^2} du.$$

Now

$$\int_{\mathbb{R}} |f|^p d\mu = \lim_{n\to\infty} \frac{1}{\sqrt{p}} \int_{-\sqrt{p}n}^{\sqrt{p}n} e^{-u^2} du = \frac{1}{\sqrt{p}} \lim_{N\to\infty} \int_{-N}^{N} e^{-u^2} du = \frac{\sqrt{\pi}}{\sqrt{p}}.$$

This shows that $f \in \mathcal{L}^p$ for all $1 \le p < \infty$. Moreover,

$$\|f\|_p = \left(\int_{\mathbb{R}} |f|^p d\mu \right)^{\frac{1}{p}} = \left(\frac{\pi}{p} \right)^{\frac{1}{2p}}.$$

2.5 a) Notice that for $x \geq 1$, $n \in \mathbb{N}$ and $1 \leq p < \infty$ we have

$$|f_n(x)|^p = \frac{n^p}{(nx^{1/3}+1)^p} \leq \frac{n^p}{(nx^{1/3})^p} = x^{-p/3}.$$

Using 2.2, we obtain

$$\int_{[1,\infty)} |f_n|^p \, d\mu \leq \int_{[1,\infty)} x^{-p/3} \, d\mu(x) \underset{\uparrow}{=} \frac{1}{p/3-1}$$

Since $p/3 > 1$.

and so $f_n \in \mathcal{L}^p$ for each $n \in \mathbb{N}$ and $p > 3$.

b) Note that in a) we actually show that there exists a $g \in \mathcal{L}^p$ ($p > 3$), namely $g(x) = x^{-1/3}$, such that $|f_n| \leq g$ for all $n \in \mathbb{N}$. Now

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{x^{1/3} + 1/n} = x^{-1/3} = g(x),$$

which shows that $(f_n)_{n=1}^{\infty}$ converges pointwise to a measurable function (namely g). Since $f_n \in \mathcal{L}^p$ for each n, we get by Proposition 2.1.4 in ELA that $[g]$ is the limit of $([f_n])_{n\in\mathbb{N}}$ in L^p.

2.8 Let $f \in \mathcal{L}^p(\mathbb{R}, \mu)$, and let $\varepsilon > 0$.

Then there exists an $N \in \mathbb{N}$ such that

$$\int_{\mathbb{R} \setminus [-N,N]} |f|^p \, d\mu < \varepsilon_1 := \frac{\varepsilon^p}{3}$$

Now since

$$\int_{[-N,N]} |f|^p \, d\mu \leq \int_{\mathbb{R}} |f|^p \, d\mu < \infty$$

we have that $f|_{[-N,N]} \in \mathcal{L}^p([-N,N], \mu)$, and so by exercise 2.7, there exists a $g \in C([-N,N])$ such that

$$\| f|_{[-N,N]} - g \|_p^p < \varepsilon_2 := \frac{\varepsilon^p}{3(2^p+1)}$$

We must now extend g to a function in $C_c(\mathbb{R})$.

Let $L_\varepsilon = \varepsilon^p / (3 \cdot 2^{p-1} \cdot (|f(-N)|^p + |f(N)|^p + 1))$.

$$\tilde{g}(x) = \begin{cases} 0 & : \quad x < -N - L_\varepsilon \\ \text{linear} & : \quad -N - L_\varepsilon \leq x < -N \\ f(x) & : \quad -N \leq x < N \\ \text{linear} & : \quad N \leq x < N + L_\varepsilon \\ 0 & : \quad N + L_\varepsilon \leq x \end{cases}$$

Here, linear means the unique line segment in the complex plane that joins the endpoints (0 and $f(-N)$ in the first case, $f(N)$ and 0 in the second case). Thus, $\tilde{g}$ is in $C_c(\mathbb{R})$.

We will now estimate $\| f - \tilde{g} \|_p$.

$$\left(\int_{-N-L_\varepsilon}^{-N} |f-\tilde{g}|^p d\mu\right)^{\frac{1}{p}} \underset{\substack{\text{triangle}\\\text{inequality}}}{\leq} \left(\int_{-N-L_\varepsilon}^{-N} |f|^p d\mu\right)^{\frac{1}{p}} + \left(\int_{-N-L_\varepsilon}^{-N} |\tilde{g}|^p d\mu\right)^{\frac{1}{p}}$$

$$\leq \left(\int_{\mathbb{R}\setminus[-N,N]} |f|^p d\mu\right)^{\frac{1}{p}} + \left(L_\varepsilon \cdot \left(\max_{-N-L_\varepsilon \leq x \leq -N} |\tilde{g}(x)|^p\right)\right)^{\frac{1}{p}}$$

$$< \varepsilon_1^{1/p} + L_\varepsilon^{1/p} |f(-N)|$$

Similarly:

$$\left(\int_{N}^{N+L_\varepsilon} |f-\tilde{g}|^p d\mu\right)^{\frac{1}{p}} \leq \left(\int_{N}^{N+L_\varepsilon} |f|^p d\mu\right)^{\frac{1}{p}} + \left(\int_{N}^{N+L_\varepsilon} |\tilde{g}|^p d\mu\right)^{\frac{1}{p}}$$

$$\leq \varepsilon_1^{1/p} + L_\varepsilon^{1/p} |f(N)|.$$

Now

$$\|f-\tilde{g}\|_p^p = \int_{[-N,N]} |f-g|^p d\mu + \int_{-N-L_\varepsilon}^{-N} |f-\tilde{g}|^p d\mu + \int_{N}^{N+L_\varepsilon} |f-\tilde{g}|^p d\mu$$

$$+ \int_{\mathbb{R}\setminus[-N-L_\varepsilon, N+L_\varepsilon]} |f-\tilde{g}|^p d\mu < \varepsilon_2 + \left(\varepsilon_1^{1/p} + L_\varepsilon^{1/p}|f(-N)|\right)^p + \left(\varepsilon_1^{1/p} + L_\varepsilon^{1/p}|f(N)|\right)^p + \varepsilon_1$$

$$\leq \varepsilon_2 + 2^{p-1}\left(\varepsilon_1 + L_\varepsilon |f(-N)|^p\right) + 2^{p-1}\left(\varepsilon_1 + L_\varepsilon |f(N)|^p\right) + \varepsilon_1$$

$$= \varepsilon_2 + L_\varepsilon 2^{p-1}\left(|f(-N)|^p + |f(N)|^p\right) + \varepsilon_1(2^p+1)$$

$$\leq \frac{\varepsilon^p}{3} + \frac{\varepsilon^p}{3} + \frac{\varepsilon^p}{3} = \varepsilon^p$$

$\uparrow$ Using $(a+b)^p \leq 2^{p-1}(a^p+b^p)$ for $p \geq 1$

It follows that

$$\|f-\tilde{g}\|_p < \varepsilon.$$

2.13 b) Since $\int_X |f|^r \, d\mu < \infty$, we have that

$$\int_X ||f|^p|^{\frac{r}{p}} \, d\mu = \int_X |f|^r \, d\mu < \infty, \quad \text{and so} \quad |f|^p \in \mathcal{L}^{r/p}.$$

Now the function constantly equal to 1 is in $\mathcal{L}^q$ for all $1 \leq q < \infty$ since $\int_X |1|^q \, d\mu = \mu(X) < \infty$, and so $1 \in \mathcal{L}^q$ for $\frac{1}{(r/p)} + \frac{1}{q} = 1$. By Hölder's inequality, we get that

$$\int_X |f|^p \, d\mu = \int_X |f|^p 1 \, d\mu \leq \| |f|^p \|_{r/p} \| 1 \|_q .$$

Now $\| |f|^p \|_{r/p} = \| f \|_p^p$, and solving for q we get

$$q = \frac{r/p}{r/p - 1} = \frac{r}{r-p} .$$

Since $\| 1 \|_q = (\mu(X))^{1/q} = \mu(X)^{1-p/r}$ we get

$$\| f \|_p = \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}} \leq \| f \|_r \left(\mu(X)^{1-p/r} \right)^{\frac{1}{p}}$$

$$= \| f \|_r \, \mu(X)^{\frac{1}{p} - \frac{1}{r}}$$

which finishes the proof.

Extra Exercise 18

a) Case 1: If $A=\emptyset$, then $\int_A \rho \, d\mu = 0$.

On the other hand, $\sum_{x\in A} \rho(x) = 0$ by definition, and so they are equal in this case.

Case 2: Let $A = \{a_1, \dots, a_n\}$ be nonempty finite. By countable additivity, we have that

$$\int_A \rho \, d\mu = \sum_{i=1}^{n} \int_{\{a_i\}} \rho \, d\mu \overset{\rho \text{ is constant on } \{a_i\}}{=} \sum_{i=1}^{n} \rho(a_i) \int_{\{a_i\}} 1 \, d\mu$$

$$= \sum_{i=1}^{n} \rho(a_i)\mu(a_i) \underset{\text{Counting measure}}{=} \sum_{i=1}^{n} \rho(a_i) \underset{\text{By definition.}}{=} \sum_{x\in A} \rho(x).$$

Case 3: Let A be infinite. Suppose first that $\rho = 1_E$ for some $E \in \mathcal{P}(X)$. Then if $|A\cap E| < \infty$:

$$\int_A \rho \, d\mu = \mu(A\cap E) = |A\cap E| = \sum_{x\in A} 1_E(x) = \sum_{x\in A} \rho(x).$$

If $|A\cap E| = \infty$ then both are infinite, as $\int_A \rho \, d\mu = \mu(A\cap E) = \infty$ and $\sum_{x\in A} \rho(x) = \infty$ since we can find finite sets $E_1 \subsetneq E_2 \subsetneq \dots \subseteq A\cap E$ so that

$$\sum_{x\in E_1} 1_E(x) < \sum_{x\in E_2} 1_E(x) < \dots .$$

Next, suppose $\rho = \sum_{j=1}^{h} a_j 1_{E_j}$ for $a_j \geq 0$ and $E_j \in \mathcal{P}(X)$.

Then

$$\int_A \rho \, d\mu \underset{\text{linearity}}{=} \sum_{j=1}^{h} a_j \int_A 1_{E_j} \, d\mu \underset{\text{already proved for } \rho = 1_E}{=} \sum_{j=1}^{h} a_j \sum_{x\in A} 1_{E_j}(x)$$

$$= \sum_{x \in A} \sum_{j=1}^{h} a_j \mathbb{1}_{E_j}(x) = \sum_{x \in A} \rho(x).$$

This shows what we want for simple functions. Now let ρ be any function on X that takes nonnegative values.

Then by definition:

$$\int_A \rho \, d\mu = \sup \left\{ \int_A s \, d\mu : 0 \le s \le \rho \, , \; s \text{ simple} \right\}$$

$$= \sup \left\{ \sum_{x \in A} s(x) : 0 \le s \le \rho , \; s \text{ simple} \right\}$$

By what we just proved.

Now since $0 \le s \le \rho$ implies $\sum_{x \in A} s(x) \le \sum_{x \in A} \rho(x)$, we have that $\int_A \rho(x) \, d\mu(x) \le \sum_{x \in A} \rho(x)$, which shows one inequality. But if $F \subseteq A$ is a finite subset, then let $s = \sum_{x \in F} \rho(x) \mathbb{1}_{\{x\}}$. This defines a simple function with $0 \le s \le \rho$, and so since $\sum_{x \in F} \rho(x) = \sum_{x \in A} s(x)$, we get

$$\sum_{x \in A} \rho(x) = \sup \left\{ \sum_{x \in F} \rho(x) : F \subseteq A \text{ finite} \right\}$$

$$\le \sup \left\{ \sum_{x \in A} s(x) : 0 \le s \le \rho \right\} = \int_A \rho \, d\mu.$$

This shows that

$$\int_A \rho \, d\mu = \sum_{x \in A} \rho(x)$$

for general $\rho : X \to \mathbb{R}_+$.

b) From the definition of the p-norm, we get

$$\|f\|_p = \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}} \underset{a)}{\overset{\uparrow}{=}} \left(\sum_{x \in X} |f(x)|^p \right)^{\frac{1}{p}}.$$

Thus $\mathcal{L}^p(X, \mathcal{A}, \mu) = \left\{ f \in \mathcal{M} : \sum_{x \in X} |f(x)|^p < \infty \right\}$

Now if $f = g$ μ-a.e. then the set

$N = \{ x \in X : f(x) \neq g(x) \}$ has measure zero.

But then $|N| = 0$ so that $N = \emptyset$, hence $f(x) = g(x)$

for all $x \in X$.

c) ~~Suppose~~ Recall that $\sum_{x \in X} |f(x)|^p < \infty$ if and only

if $\sum_{x \in X \setminus F} |f(x)|^p$ for any (hence all) finite subsets

$F \subseteq X$. Now suppose $f \in \mathcal{L}^p(X)$. Since

$\sum_{x \in X} |f(x)|^p < \infty$, there exists a finite subset F

of X such that $|f(x)|^p < 1$ for all $x \in X \setminus F$,

$\Rightarrow |f(x)| < 1$ for $x \in X \setminus F$.

If $1 \leq p \leq r < \infty$ then

$$|f(x)|^p \geq |f(x)|^r \quad \text{for } x \in X \setminus F,$$

and so

$$\sum_{x \in X \setminus F} |f(x)|^r \leq \sum_{x \in X \setminus F} |f(x)|^p < \infty.$$

By the first remark, we get that $f \in \ell^r(X)$.

Also, the fact that $|f(x)| < 1$ for all $x \in X \setminus F$

shows that

$$|f(x)| \leq \max \left\{ \max_{y \in F} |f(y)|, 1 \right\} \text{ and so } f \in \ell^\infty(X)$$

as well.