MAT3400/4400 - Spring 19 - Exercises for Monday, Mars 18

- From the exercises in the Notes on ELA: 2.2, 2.3, 2.5, 2.8, 2.13 b).

In addition, Exercise 2.1 is a routine exercise that everybody should do. But it will not be discussed in class unless there is time for it.

Extra Exercise 18

Let X be a nonempty set, set $\mathcal{A} = \mathcal{P}(X)$ (the σ-algebra consisting of all subsets of X), and let μ denote the counting measure on $\mathcal{A}$. Note that in this case, the space $\mathcal{M}$ of all complex-valued measurable functions on X consists of all complex-valued functions on X.

a) Let $\rho : X \to \mathbb{R}_+$ and $A \in \mathcal{P}(X)$. Show that

$$\int_A \rho \, d\mu = \sum_{x \in A} \rho(x)$$

where the sum is defined as in Extra Exercise 1, cf. the exercise set for Jan. 21. Note that this shows that the measure $\mu_\rho : \mathcal{P}(X) \to [0, \infty]$, given by

$$\mu_\rho(A) = \sum_{x \in A} \rho(x) \, , \quad A \in \mathcal{P}(X) \, ,$$

which was introduced in this exercise, satisfies that $\mu_\rho(A) = \int_A \rho \, d\mu$ for all $A \in \mathcal{P}(X)$.

b) Let $p \in [1, \infty)$ and let $f \in \mathcal{M}$. Deduce from a) that $\|f\|_p = \big(\sum_{x \in X} |f(x)|^p\big)^{1/p}$. Conclude that

$$\mathcal{L}^p(X, \mathcal{A}, \mu) = \Big\{ f \in \mathcal{M} : \sum_{x \in X} |f(x)|^p < \infty \Big\}.$$

Verify also that if $f, g \in \mathcal{L}^p(X, \mathcal{A}, \mu)$, then $f = g$ μ-a.e. if and only if $f = g$. Note that this implies that $L^p(X, \mathcal{A}, \mu) = \mathcal{L}^p(X, \mathcal{A}, \mu)$. As is common, we set $\ell^p(X) := \mathcal{L}^p(X, \mathcal{A}, \mu)$.

c) Let $1 \leq p \leq r < \infty$ and let $\ell^\infty(X)$ denote the space of all bounded complex-valued functions on X. Show that

$$\ell^p(X) \subseteq \ell^r(X) \subseteq \ell^\infty(X).$$