MAT3400/4400 Spring 2019 Exercises 25.03 SOLUTIONS

2.10 Suppose first that $f \in \mathcal{L}^\infty$. Then there exists $M \geq 0$ such that $|f| \leq M$ μ-a.e. Define $g : X \to \mathbb{C}$ by

$$g(x) = \begin{cases} f(x) \;; & |f(x)| \leq M \\ M \;; & |f(x)| > M \end{cases} \quad (*) .$$

Then g is bounded, and if $x \in X$ is such that $f(x) \neq g(x)$ then $|f(x)| > M$, and so $f = g$ μ-a.e.

Conversely, if $f = g$ μ-a.e. where $g : X \to \mathbb{C}$ is a bounded function, then let $N \geq 0$ be a constant such that $|g(x)| \leq N$ for all $x \in X$. If $x \in X$ is such that $|f(x)| > N$ then $g(x) \neq f(x)$, and so $|f| \leq N$ μ-a.e.

In either case, if $M \geq 0$ is such that $|f| \leq M$ μ-a.e., then g as in $(*)$ satisfies $M = \|g\|_u$. On the other hand, if $f = g$ μ-a.e. for some bounded $g : X \to \mathbb{C}$, then since $|g(x)| \leq N$ for all $x \in X$ with $N = \|g\|_u$, we get from the above that $|f| \leq \|g\|_u$ μ-a.e. Hence, we have that

$$\{ M \geq 0 : |f| \leq M \ \mu\text{-a.e.} \} = \{ \|g\|_u : g = f \ \mu\text{-a.e.}, \ g \text{ bounded} \}$$

and so

$$\underset{\substack{\uparrow \\ \text{by def}}}{\|f\|_\infty} = \inf \{ M \geq 0 : |f| \leq M \ \mu\text{-a.e.} \}$$

$$= \inf \{ \|g\|_u : g = f \ \mu\text{-a.e.}, \ g \text{ bounded} \}.$$

2.11 Suppose $(X, \mathcal{A}, \mu)$ is a measure space and $E \in \mathcal{A}$ with $\mu(E^c) = 0$. Suppose $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}^\infty(X, \mathcal{A}, \mu)$, $f \in \mathcal{L}^\infty(X, \mathcal{A}, \mu)$ and that $(f_n)_n$ converges uniformly to f on E. This means that

$$\|(f_n - f)\mathbb{1}_E\|_u \to 0 \quad \text{as} \quad n \to \infty.$$

For every $n \in \mathbb{N}$, we have that

$$f_n - f = (f_n - f)\mathbb{1}_E \quad \mu\text{-a.e., since } \mu(E^c) = 0$$

$$(f_n - f)\mathbb{1}_E \leq \|(f_n - f)\mathbb{1}_E\|_u$$

which shows that $f_n - f$ is equal μ-a.e. to the bounded function $(f_n - f)\mathbb{1}_E$. Hence, by 2.10, we get that

$$\|f_n - f\|_\infty = \inf\{\|g\|_u : g = f_n - f \ \mu\text{-a.e. \& } g \text{ bounded}\}$$
$$\leq \|(f_n - f)\mathbb{1}_E\|_u \to 0$$

which proves that $(f_n)_{n\in\mathbb{N}}$ converges to f in the $\mathcal{L}^\infty$-norm.

2.13 a) Let $f \in \mathcal{L}^\infty$. By 2.10, $f = g$ μ-a.e. for some bounded function $g : X \to \mathbb{C}$. But then for any $1 \leq p < \infty$:

$$\int_X |f|^p \, d\mu = \int_X |g|^p \, d\mu \leq \int_X \|g\|_u^p \, d\mu = \|g\|_u^p \, \mu(X) < \infty$$

This shows that $f \in \mathcal{L}^p$.

c) Let $f, g : \mathbb{R} \to \mathbb{C}$ be defined by

$$f(x) = \begin{cases} 1/x & ; \quad x \geq 1 \\ 0 & ; \quad x < 1 \end{cases}$$

$$g(x) = 1 \quad \text{for all } x \in \mathbb{R}.$$

Then

$$\int_{\mathbb{R}} |f|^2 \, d\mu = \int_{[1,\infty)} \frac{1}{x^2} \, d\mu(x) \underset{\substack{\uparrow \\ \text{using the} \\ \text{Riemann integral} \\ \text{trick}}}{=} \lim_{n\to\infty} \int_1^n \frac{1}{x^2} \, dx = 1$$

while

$$\int_{\mathbb{R}} |f| \, d\mu = \int_{[1,\infty)} \frac{1}{x} \, dx = \lim_{n\to\infty} \int_1^n \frac{1}{x} \, dx = \infty.$$

Also,

$$\int_{\mathbb{R}} |g|^2 \, d\mu = \int_{\mathbb{R}} 1 \, d\mu = \mu(\mathbb{R}) = \infty$$

while

$$\operatorname*{ess\,sup}_{x \in \mathbb{R}} |g(x)| = \sup_{x \in \mathbb{R}} 1 = 1.$$

This shows that

$$f \in \mathcal{L}^2 \quad f \notin \mathcal{L}^1$$
$$g \in \mathcal{L}^\infty \quad g \notin \mathcal{L}^2$$

2.14 Let $f \in \mathcal{L}^\infty$. By 2.10, since we are working with equivalence classes in L^∞ in the end, we can assume without loss of generality that f is bounded.

Suppose first that f is real-valued. Write $\mathbb{R}$, for any fixed $n \in \mathbb{N}$, as the disjoint union

$$\mathbb{R} = \bigcup_{k \in \mathbb{Z}} \left[\frac{k}{n}, \frac{k+1}{n}\right). \quad (*)$$

Define $h_n : X \to \mathbb{C}$ by

$$h_n(x) = \frac{k}{n} \qquad \text{if} \qquad f(x) \in \left[\frac{k}{n}, \frac{k+1}{n}\right) \text{ for some (unique) } k \in \mathbb{Z}.$$

This is well-defined because of $(*)$.

Moreover, since f is bounded, we have that ~~for~~ there exists some $N \in \mathbb{N}$ with $f(x) \in \bigcup_{|k| \leq N} \left[\frac{k}{n}, \frac{k+1}{n}\right)$ for all $x \in X$.

Hence h_n has finite range (contained in the set $\left\{-\frac{N}{n}, \frac{-N+1}{n}, \ldots, 0, \frac{1}{n}, \ldots, \frac{N}{n}\right\}$) and so h_n is simple.

Moreover,

$$\|f - h_n\|_\infty = \|f - h_n\|_u = \sup_{x \in X} |f(x) - h_n(x)|$$

$$= \sup_{|k| \leq N} \sup_{x \in X;\ f(x) \in [\frac{k}{n}, \frac{k+1}{n})} |f(x) - h_n(x)|$$

$$= \sup_{|k| \leq N} \sup_{x \in X;\ f(x) \in [\frac{k}{n}, \frac{k+1}{n})} \left|f(x) - \frac{k}{n}\right| \leq \left|\frac{k+1}{n} - \frac{k}{n}\right| = \frac{1}{n} \xrightarrow{n \to \infty} 0.$$

This shows that $(h_n)_{n=1}^\infty$ is a sequence of simple functions converging ~~pointwise to~~ in L^∞ to f. If f is complex-valued, do the usual trick of approximating the real and imaginary part in the L^∞-norm by simple functions.

3.2. a) Let $x \in X \setminus M$. By corollary 3.1.5 in ELA, M is closed in X (being finite-dimensional) and so $X \setminus M$ is open. Hence there exists $r > 0$ such that $B_r(x) \subseteq X \setminus M$. In other words, $\|x - m\| \geq r$ for $m \in M$, and so $\inf_{m \in M} \|x - m\| \geq r > 0$.

b) Let $x \in X \setminus M$, $d := \inf_{m \in M} \|x - m\| > 0$. By definition of infimum, we can find $m_0 \in M$ with $\|x - m_0\| < 2d$. Let

$$y_0 = \frac{x - m_0}{\|x - m_0\|}. \quad \text{Then} \quad \|y_0\| = 1$$

For all $m \in M$:

$$y_0 - m = \frac{1}{\|x - m_0\|} \left(x - (m_0 + \|x - m_0\| m) \right)$$

Since $m_0 + \|x - m_0\| m \in M$, we get that

$$\|y_0 - m\| \geq \frac{1}{\|x - m_0\|} d \geq \frac{d}{2d} = \frac{1}{2}.$$

This shows that y_0 has the properties we are looking for.

c) Suppose X is infinite-dimensional. Pick any $y_1 \in X_1$. Since $M_1 = \text{span}\{y_1\}$ is finite-dimensional, $M_1 \subsetneq X$, and so by b) there is $y_2 \in X_1$ with $\|y_2 - y_1\| \geq \frac{1}{2}$. Now let $M_2 = \text{span}\{y_1, y_2\}$. Then M_2 is finite-dimensional and so $M_2 \subsetneq X$, so by b) there exists $y_3 \in X_1$ with $\|y_3 - y_k\| \geq \frac{1}{2}$ for $k = 1, 2$. Continuing like this, we obtain a sequence $(y_n)_{n=1}^{\infty}$ in X_1 with $\|y_k - y_l\| \geq \frac{1}{2}$ for $k \neq l$. This sequence has no convergent subsequences, which shows that X_1 is noncompact.

3.3 a) Let $n \in \mathbb{N}$, and $f_1, \dots, f_n \in X$. Then

$$N := \max\{k \in \mathbb{N} : \exists\, i \in \{1, \dots, n\} \text{ with } f_i(k) \neq 0\} < \infty.$$

Now if $f \in \operatorname{span}\{f_1, \dots, f_n\}$ then $f(N+1) = 0$. Hence $g : \mathbb{N} \to \mathbb{C}$ defined by $g(k) = \begin{cases} 1; & k = N+1 \\ 0; & \text{otherwise} \end{cases}$

is in X and satisfies $g(N+1) = 1$, which shows that $g \notin \operatorname{span}\{f_1, \dots, f_n\}$. We conclude that X is infinite-dimensional.

b) Let $f_n(k) = \begin{cases} 1; & k \leq n \\ 0; & k > n. \end{cases}$

Then $L(f_n) = n$ while $\|f_n\|_u = 1$. If L was bounded then this would imply the existence of $C \geq 0$ such that

$$|L(f)| \leq C\|f\|_u \quad \text{for all } f \in X,$$

and so

$$n = |L(f_n)| \leq C\|f\|_u = C \quad \text{for all } n \in \mathbb{N}$$

which is impossible. Hence L is unbounded.

Also, let g_n be defined by $g_n(1) = 1$ and $g_n(2) = g_n(3) = \dots = g_n(n+1) = -\frac{1}{n}$, $g_n(n+2) = g_n(n+3) = \dots = 0$.

Then $L(g_n) = 1 - \underbrace{\left(\frac{1}{n} + \dots + \frac{1}{n}\right)}_{n} = 0$ so $g_n \in \operatorname{Ker}(L)$.

Let $g(k) = \begin{cases} 1; & k = 1 \\ 0; & k > 1. \end{cases}$ Then $g \in X$,

$\|g - g_n\|_u = \frac{1}{n} \to 0$ but $L(g) = 1$. This shows that $\operatorname{Ker}(L)$ is not closed in X.

3.4 a) We recognize $P_{\mathbb{R}}$ as the subspace of $C([0,1])$ consisting of polynomial functions. Since the norm on $C([0,1])$ is the supremum norm $\|f\| = \sup_{t \in [0,1]} |f(t)|$, it follows that, this norm restricted to $P_{\mathbb{R}}$ gives a norm.

b) Note that $D(x^n) = nx^{n-1}$,

$$\|nx^{n-1}\|_u = \sup_{0 \le x \le 1} |nx^{n-1}| = n$$

while $\|x^n\|_u = 1$. Hence D is unbounded, as there is not any constant $C \geq 0$ such that

$$n = \|D(x^n)\| \leq C\|x^n\| = C$$

for all $n \in \mathbb{N}$.

By Prop 3.1.7, in ELA, an operator on a finite-dimensional normed space is always bounded. Hence, $P_{\mathbb{R}}$ cannot be finite-dimensional.