MAT4400 Spring 2019 Exercise SOLUTIONS April 1st

3.7 We have to show a number of things in this exercise:

i) <u>M_1 is closed</u>:

Suppose $(f_k)_{k=1}^{\infty} \subseteq M_1$ with $(f_k)_k \to f \in X$ in the ℓ^1-norm. Then

$$\sum_{n\in\mathbb{N}} |f(2n) - nf(2n-1)|$$

$$\leq \sum_{n\in\mathbb{N}} |f(2n) - f_k(2n)| + \sum_{n\in\mathbb{N}} |f_k(2n) - nf_k(2n-1)|$$

$$+ \sum_{n\in\mathbb{N}} |nf_k(2n-1) - nf(2n-1)|$$

$$\leq \|f - f_k\|_1 + 0 + C\|f_k - f\|_1$$

where C is the biggest n such that for all $k \in \mathbb{N}$, $|nf_k(2n-1) - nf(2n-1)| \neq 0$ (note that such an n exists since $f_k, f \in X$ and $f_k \to f$ in ℓ^1).

Taking the limit as $k \to \infty$, we get that

$$\sum_{n\in\mathbb{N}} |f(2n) - nf(2n-1)| = 0 \quad \text{or} \quad f(2n) = nf(2n-1)$$

for all $n \in \mathbb{N}$.

(ii) M_2 is closed:

Let $T: X \to X$ be given by $(Tf)(n) = f(2n-1)$ for $f \in X$, $n \in \mathbb{N}$. Then T is linear, and

$$\|Tf\|_1 = \sum_{n \in \mathbb{N}} |f(2n-1)| \leq \sum_{n \in \mathbb{N}} |f(n)| = \|f\|_1$$

shows that T is bounded. Hence T is continuous by Prop 1.1.1 in ELA, and so $\operatorname{Ker} T = T^{-1}(\{0\}) = M_2$ is closed.

(iii) $M_1 \cap M_2 = \{0\}$:

Suppose $f \in M_1 \cap M_2$. Then $f(2n-1) = 0$ for all n since $f \in M_2$, but then $f(2n) = n f(2n-1) = 0$ for all n since $f \in M_1$. Since all natural numbers are either of the form $2n-1$ or $2n$ for some $n \in \mathbb{N}$, we have that $f = 0$.

(iv) $M_1 + M_2 = X$:

Let $f \in X$. Define $f_1, f_2 : \mathbb{N} \to \mathbb{C}$ by

$$f_1(n) = \begin{cases} f(n) & ; \ n \text{ odd} \\ \frac{n}{2} f(n-1) & ; \ n \text{ even} \end{cases} \qquad (*)$$

$$f_2(n) = \begin{cases} 0 & ; \ n \text{ odd} \\ f(n) - \frac{n}{2} f(n-1) & ; \ n \text{ even.} \end{cases}$$

From the definitions it is clear that $f_1 \in M_1$, $f_2 \in M_2$ and $f_1 + f_2 = f$.

v) Projection of X down to M_1 along M_2 is unbounded!

Let $P: X \to X$, $P(f) = f_1$ as in $(*)$, i.e. P is the projection down to M_1 along M_2.

Define

$$g_k(n) = \begin{cases} 1 & ; \quad n = 2k-1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Then $\|g_k\|_1 = 1$ while

$$\|P(g_k)\|_1 = \sum_{n\in\mathbb{N}} |P(g_k)(2n-1)| + \sum_{n\in\mathbb{N}} |P(g_k)(2n)|$$

$$= \sum_{n\in\mathbb{N}} |g_k(2n-1)| + \sum_{n\in\mathbb{N}} |n\, g_k(2n-1)|$$

$$= \sum_{n\in\mathbb{N}} (n+1)\, |g_k(2n-1)| = k+1 \to \infty.$$

This shows that P is unbounded.

3.9 Suppose $X = M_1 \dot{+} M_2$ and that M_2 is T-invariant.

Now $M_1 = \operatorname{Ker}(I-T) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 : \begin{pmatrix} x+iy \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \right\}$
$= \mathbb{C}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

In order for $X = M_1 \dot{+} M_2$ to hold true, we need M_2 to be a 1-dimensional subspace of $\mathbb{C}^2$, and so $M_2 = \mathbb{C}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ for some $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{C}^2$ with $y_0 \neq 0$ (otherwise $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in M_1$).

For M_2 to be T-invariant, we need $T\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 + iy_0 \\ y_0 \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ for some $\lambda \in \mathbb{C}$, and so $y_0 = \lambda y_0$ gives $\lambda = 1$ since $y_0 \neq 0$. But then $x_0 + iy_0 = x_0$ which gives $y_0 = 0$, a contradiction. Hence no such subspace M_2 exists.

3.10 a) $S^2 = (P_1 - P_2)^2 = P_1^2 - P_1 P_2 - P_2 P_1 + P_2^2$

$= P_1 + P_2 = I.$

↑

PROP 3.2.6.
in ELA

Furthermore, we have

$S = P_1 - P_2 = (I - P_2) - P_2 = I - 2P_2 = P_1 - (I - P_1) = 2P_1 - I,$

and so

$$P_1 = \tfrac{1}{2}(I+S)$$
$$P_2 = \tfrac{1}{2}(I-S).$$

Thus $M_1 \underset{\uparrow}{=} P_1(X) = \text{Ker}(P_2) = \text{Ker}\left(\tfrac{1}{2}(I-S)\right) = \text{Ker}(I-S)$

3.2.6.

$M_2 \underset{\uparrow}{=} \text{Ker}(P_1) = \text{Ker}\left(\tfrac{1}{2}(I+S)\right) = \text{Ker}(I+S).$

3.2.6.

b) Define $P := \tfrac{1}{2}(I+S)$. Then $I - P = \tfrac{1}{2}(I-S)$, and

$P^2 = \tfrac{1}{4}(I^2 + IS + SI + S^2) = \tfrac{1}{4}(2I + 2S) = P$

and so by ELA: PROP 3.2.8,

$$X = P(X) \dot{+} \text{Ker}(P)$$

with $P(X) = \text{Ker}(I-P)$, $(I-P)(X) = \text{Ker}(P)$ and P being the projection from X on $P(X)$ along $\text{Ker}(P)$.

But $\text{Ker}(I-S) = \text{Ker}\left(\tfrac{1}{2}(I-S)\right) = \text{Ker}(I-P) = P(X)$

and $\text{Ker}(I+S) = \text{Ker}\left(\tfrac{1}{2}(I+S)\right) = \text{Ker}(P)$,

and $S = 2\,\tfrac{1}{2}(I+S) - I = 2P - I$ which shows that S is the symmetry through $\text{Ker}(I-S)$ along $\text{Ker}(I+S)$.

To show that S is decomposable w.r.t. $X = Ker(I-S) \dot{+} Ker(I+S)$, we show that S commutes with P and appeal to ELA: Prop 3.2.20:

$$PS = \frac{1}{2}(I+S)S = \frac{1}{2}(IS+S^2) = \frac{1}{2}(S+I) = P$$

$$SP = \frac{1}{2}(SI+S^2) = \frac{1}{2}(S+I) = P$$

and so S and P commute.

c) Since $S \in \mathcal{B}(X)$, S is continuous. Since I is also continuous, we have that $I-S$ and $I+S$ are continuous, and so $Ker(I-S) = (I-S)^{-1}(\{0\})$ and $Ker(I+S) = (I+S)^{-1}(\{0\})$ are closed. Hence

$$X = Ker(I-S) \oplus Ker(I+S).$$

d) $\underline{S \text{ is bounded:}}$

$$\|S(f)\|_u = \max_{t\in[-a,a]} |S(f)(t)| = \max_{t\in[-a,a]} |f(-t)|$$

$$= \max_{t\in[-a,a]} |f(t)| = \|f\|_u. \quad (\text{for all } f \in X)$$

$\underline{S^2 = I:}$

$$S^2(f)(t) = S(f)(-t) = f(-(-t)) = f(t). \quad (\forall f \in X)$$

From b) and c) we get $X = Ker(I-S) \oplus Ker(I+S)$.

But $Ker(I-S) = X_{even}$

$Ker(I+S) = X_{odd}$.

3.11 a) Let $\{\lambda_1, \lambda_2, \lambda_3\}$ be the eigenvalues of the 3x3 orthogonal matrix $U \neq I$ with $\det U = 1$, counting multiplicity. We need the following facts from linear algebra:

1) An orthogonal matrix is diagonalizable, and so $\mathbb{C}^3$ has a basis consisting of eigenvectors of U, and the multiplicity of λ_j is equal to $\dim(E^U_{\lambda_j})$.

2) $|\lambda_j| = 1$ for $j = 1, 2, 3$ (since U is orthogonal)

3) $\lambda_1 \lambda_2 \lambda_3 = \det U = 1$ (the first equality holds for any matrix)

Now we consider two cases:

Case I: All eigenvalues are real. Then by 2), $\lambda_j = \pm 1$ for each j. If $\lambda_1 = \lambda_2 = \lambda_3 = 1$ then $\dim(E^U_1) = 3$ and so $E^U_1 = \mathbb{R}^3$. If $\{e_1, e_2, e_3\}$ is a basis consisting of eigenvectors of U, then $Ue_j = e_j$ for each j and so $U = I$, a contradiction.
Hence $\lambda_j = -1$ for some j, say $\lambda_1 = -1$. If $\lambda_2 = \lambda_3 = -1$ then $\lambda_1 \lambda_2 \lambda_3 = -1$ which contradicts 3). Thus we have, say $\lambda_1 = \lambda_2 = -1$ while $\lambda_3 = 1$, and so $\dim(E^U_1) = (\text{multiplicity of } 1) = 1$.

Case II: There are complex eigenvalues, say λ_1 is complex (not real). Then $\overline{\lambda_1}$ is also an eigenvalue since U is a real matrix, say $\lambda_2 = \overline{\lambda_1}$. Now 3) gives
$1 = \lambda_1 \lambda_2 \lambda_3 = |\lambda_1|^2 \lambda_3 = \lambda_3$, and so $\lambda_3 = 1$. Also, By 1) $\dim E^U_1 = \text{mult. of } 1 = 1$.

b) We show that both M and N are invariant under U, and appeal to ELA: Prop 3.2.20.

Now $M = E_1^U$ is an eigenspace of U and is therefore invariant under U.

Let us show that $N := M^\perp$ is invariant under U. Suppose $x \in N$, so that $\langle x, y \rangle = 0$ for all $y \in M$, i.e. $\langle x, y \rangle = 0$ for all $y \in \mathbb{R}^3$ with $Uy = y$ (equivalently $U^{-1}y = y$).

But then, for $y \in M$, we have

$$\langle Ux, y \rangle \underset{\uparrow}{=} \langle U^{-1}Ux, U^{-1}y \rangle = \langle x, U^{-1}y \rangle \underset{\uparrow}{=} \langle x, y \rangle \underset{\uparrow}{=} 0$$

since U^{-1} is also orthogonal, hence it preserves the Euclidean inner product.

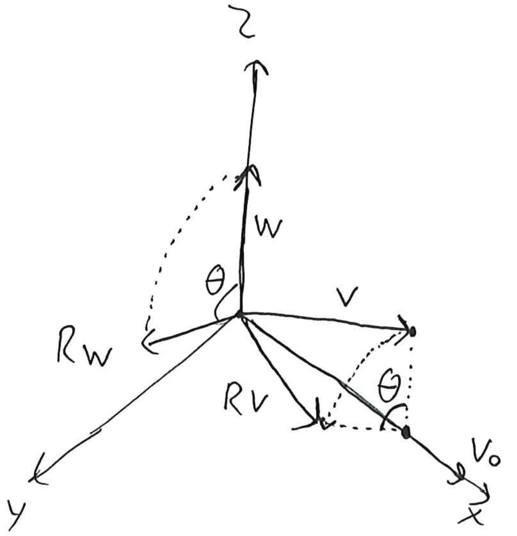
since $y \in M$

since $x \in M^\perp$

This shows that $Ux \in N$ as well, and so $M^\perp$ is invariant under U.

c) Since the matrix U' of R' is a matrix w.r.t. an orthonormal basis of two vectors, it is a 2×2 orthogonal matrix. ~~It is well-known~~ Now we know from a) that the eigenvalues of U' are either $\{1,1\}$ or $\{\lambda, \bar{\lambda}\}$ for $\lambda \in \mathbb{C}$. In either case, $\det U' = 1$, and so U' is a rotation matrix (it is well-known that 2×2 orthogonal matrices with determinant 1 are precisely the rotation matrices).

d) R (or equivalently its matrix U) acts as the identity on the 1-dimensional subspace E_1^U, and as a rotation matrix on the orthogonal complement of E_1^U. If v_0 is a spanning vector for E_1^U, then we can visualize this as vectors being rotated around the subspace spanned by v_0: In the illustration, $v_0 = (1, 0, 0)$.



3.14. Let $M_1 = M_2 = \mathbb{R} \times \{0\}$ and $M_3 = \{0\} \times \mathbb{R}$.

Then $M_1 + M_2 + M_3 = \mathbb{R}^2$, $M_1 \cap M_2 \cap M_3 = \{(0,0)\}$.

But $(1,0) \in M_1$, $(-1,0) \in M_2$ and $(0,0) \in M_3$ satisfy

$$(1,0) + (-1,0) + (0,0) = (0,0)$$

while not being all equal to $(0,0)$. This shows that we do not have that X is the direct sum of M_1, M_2 and M_3.

3.15 Suppose I_0 extends to a bounded linear map $\tilde{I} : X \to X_0$. Since X_0 is not complete, it is not closed in X, and so it must be a proper subspace of X. Pick $x \in X \setminus X_0$. Let $(x_n)_{n=1}^{\infty}$ be a sequence in X_0 with $(x_n)_n \to x$ (X_0 is dense in X).

Then $\tilde{I}(x) \underset{\substack{\uparrow \\ \text{since } \tilde{I} \\ \text{is bounded,} \\ \text{it is continuous}}}{=} \lim_{n\to\infty} \tilde{I}(x_n) \underset{\substack{\uparrow \\ \text{Since } \tilde{I} \\ \text{extends } I_0}}{=} \lim_{n\to\infty} I_0(x_n) = \lim_{n\to\infty} x_n = x$.

But the codomain of $\tilde{I}$ is X_0 and $x \in X \setminus X_0$, so this is a contradiction. Hence I_0 does not extend to a bounded linear map $X \to X_0$.

3.18 a) We show that $T_K(f)$ is continuous on $[a,b]$.

Let $\varepsilon>0$. If $f(t)=0$ for all $t\in[a,b]$ then $T_K(f)(s)=0$ for all $s\in[a,b]$ which is continuous, so assume that f is not identically zero.

Then by continuity of f and compactness of $[a,b]$, f is bounded on $[a,b]$, and moreover $0<\|f\|_u<\infty$.

$\left[\ \|f\|_u = \sup_{t\in[a,b]} |f(t)|\ \right]$.

Moreover, K is continuous on $[a,b]\times[a,b]$ which is compact, which means that K is uniformly continuous there. Hence, there exists $\delta>0$ such that $\forall s,r,t\in\mathbb{R}$:

$$|s-r|<\delta \implies |K(s,t)-K(r,t)|<\frac{\varepsilon}{(b-a)\|f\|_u}.$$

Now if $|s-r|<\delta$, then

$$|T_K(f)(s)-T_K(f)(r)| = \left|\int_a^b (K(s,t)-K(r,t))f(t)\,dt\right|$$

$$\leq \int_a^b |K(s,t)-K(r,t)|\,|f(t)|\,dt \leq \|f\|_u \int_a^b |K(s,t)-K(r,t)|\,dt$$

$$\leq \|f\|_u \int_a^b \frac{\varepsilon}{(b-a)\|f\|_u}\,dt = \frac{b-a}{b-a}\varepsilon = \varepsilon.$$

This shows that $T_K(f)$ is continuous

Moreover,

$$\|T_K f\|_2^2 = \int_a^b \left| \int_a^b K(s,t)\, f(t)\, dt \right|^2 ds$$

$$= \int_a^b \left| \int_a^b K(s,t)\, \overline{\bar{f}(t)}\, dt \right|^2 ds$$

Cauchy-Schwarz

$$\leq \int_a^b \left(\int_a^b |K(s,t)|^2\, dt \right)^{\cancel{\frac{1}{2}}} \left(\int_a^b |\bar{f}(t)|^2\, dt \right) ds$$

$$= \left(\int_a^b \int_a^b |K(s,t)|^2\, ds\, dt \right) \|f\|_2^2 .$$