MAT 3400/4400 Exercises SOLUTIONS April 8
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Exercise 4.3 Since $S \subseteq M$, we get $S^\perp \supseteq M^\perp$.

To show that $S^\perp \subseteq M^\perp$, suppose $x \in S^\perp$. We prove that $x \in M^\perp$ in three steps:

Step 1: Let $y \in S$. Then $\langle x, y \rangle = 0$ since $x \in S^\perp$.

Step 2: Let $y \in \operatorname{Span}(S)$, say $y = \sum_{j=1}^{k} \lambda_j s_j$

where $\lambda_j \in \mathbb{C}$, $s_j \in S$ for each j.

Then $\langle x, y \rangle = \sum_{j=1}^{k} \bar{\lambda}_j \langle x, s_j \rangle \underset{\text{By step 1.}}{=} \sum_{j=1}^{k} \bar{\lambda}_j \cdot 0 = 0$

Step 3: Let $y \in M$, say $y = \lim_{n \to \infty} y_n$ where $y_n \in \operatorname{Span}(S)$ for each n.

Then:

$$\langle x, y \rangle \underset{\substack{\text{continuity} \\ \text{of inner product}}}{=} \lim_{n \to \infty} \langle x, y_n \rangle = \lim_{n \to \infty} 0 = 0.$$

This shows that $x \in M^\perp$.

By Theorem 4.1.2, we have that

$$(S^\perp)^\perp = (M^\perp)^\perp = M.$$

Also, if N is a subspace, then

$$(N^\perp)^\perp = \overline{\operatorname{Span} N} = \bar{N}.$$

Exercise 4.5 Let $P \in \mathcal{B}(H)$, $P^2 = P$, $\|P\| = 1$.

We begin by showing that P is self-adjoint.

By Theorem 4.3.2 iv) we get $\|P^*\| = \|P\| = 1$.

Note also that $\langle Px, P^*x \rangle = \langle PPx, x \rangle = \langle Px, x \rangle$ for every $x \in H$. Hence

$$\|Px - P^*x\| = \|Px\|^2 + \|P^*x\|^2 - 2\mathrm{Re}(\langle Px, P^*x \rangle)$$

$$\leq \|Px\|^2 + \|P^*\| \|x\|^2 - 2\mathrm{Re}(\langle Px, P^*x \rangle)$$

$$= \|Px\|^2 + \|x\|^2 - 2\mathrm{Re}(\langle Px, x \rangle)$$

$$= \|Px - x\|^2.$$

Now let $y \in H$, and set $x = Py$. Then $Px = P^2y = Py = x$, and so $\|Px - P^*x\|^2 \leq \|Px - x\|^2 = 0$, implying that $Px = P^*x$. But then $P^2y = P^*Py$, i.e. $Py = P^*Py$ for all $y \in H$, and so $P = P^*P$. Taking adjoints and using Theorem 4.3.2. iii), we get $P^* = (P^*P)^* = P^*((P^*)^*) = P^*P = P$.

Hence $P = P^*$.

By the hint, we have that $H = P(H) \oplus \mathrm{Ker}\, P$.

By Proposition 4.3.8, $P(H)^{\perp} = \mathrm{Ker}(P^*) = \mathrm{Ker}(P)$

Since $P = P^*$.

Hence $$H = P(H) \oplus P(H)^{\perp},$$

which shows that P is the orthogonal projection of H onto $P(H)$.

Exercise 4.7. a) Linearity in the first argument:

Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in H$, $\lambda, \mu \in \mathbb{F}$.

Then

$$\langle \lambda(x_1, x_2) + \mu(y_1, y_2), (z_1, z_2) \rangle$$

$$= \langle (\lambda x_1 + \mu y_1, \lambda x_2 + \mu y_2), (z_1, z_2) \rangle$$

$$= \langle (\lambda x_1 + \mu y_1), z_1 \rangle + \langle (\lambda x_2 + \mu y_2), z_2 \rangle$$

$$= \lambda \langle x_1, z_1 \rangle + \mu \langle y_1, z_1 \rangle + \lambda \langle x_2, z_2 \rangle + \mu \langle y_2, z_2 \rangle$$

$$= \lambda(\langle x_1, z_1 \rangle + \langle x_2, z_2 \rangle) + \mu(\langle y_1, z_1 \rangle + \langle y_2, z_2 \rangle)$$

$$= \lambda \langle (x_1, x_2), (z_1, z_2) \rangle + \mu \langle (y_1, y_2), (z_1, z_2) \rangle.$$

Conjugate-symmetry:

$$\overline{\langle (x_1, x_2), (y_1, y_2) \rangle} = \overline{\langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle}$$

$$= \overline{\langle x_1, y_1 \rangle} + \overline{\langle x_2, y_2 \rangle} = \langle y_1, x_1 \rangle + \langle y_2, x_2 \rangle$$

$$= \langle (y_1, y_2), (x_1, x_2) \rangle.$$

Positive definiteness:

$$\langle (x_1, x_2), (x_1, x_2) \rangle = \langle x_1, x_1 \rangle + \langle x_2, x_2 \rangle \geq 0$$

Since both $\langle x_1, x_1 \rangle \geq 0$ and $\langle x_2, x_2 \rangle \geq 0$.

If $\langle (x_1, x_2), (x_1, x_2) \rangle = 0$ then $\langle x_1, x_1 \rangle = 0 = \langle x_2, x_2 \rangle$

which implies $x_1 = 0$, $x_2 = 0$, i.e. $(x_1, x_2) = (0, 0)$.

Finally, we must show that H is complete in the norm

$$\|(x_1, x_2)\| = (\langle (x_1,x_2),(x_1,x_2)\rangle)^{\frac{1}{2}}$$
$$= (\langle x_1,x_1\rangle + \langle x_2,x_2\rangle)^{\frac{1}{2}} = (\|x_1\|^2 + \|x_2\|^2)^{\frac{1}{2}}.$$

Suppose $(x_1^n, x_2^n)_{n=1}^{\infty}$ is a Cauchy sequence in H. Then for every $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that

$$m,n \geq N \Rightarrow (\|(x_1^m, x_2^m) - (x_1^n, x_2^n)\|$$
$$= (\|x_1^m - x_1^n\|^2 + \|x_2^m - x_2^n\|^2)^{1/2} < \varepsilon.$$

Since
$$\|x_1^m - x_1^n\| \leq \|(x_1^m, x_2^m) - (x_1^n, x_2^n)\| < \varepsilon$$
$$\|x_2^m - x_2^n\| \leq \|(x_1^m, x_2^m) - (x_1^n, x_2^n)\| < \varepsilon$$
for $m,n\geq N$, we obtain that

$(x_1^n)_{n=1}^{\infty}$ is a Cauchy sequence in H_1

$(x_2^n)_{n=1}^{\infty}$ — " — H_2.

Since H_1 and H_2 are complete, we find limits $x_1 = \lim_{n\to\infty} x_1^n$, $x_2 = \lim_{n\to\infty} x_2^n$. We will show that $\lim_{n\to\infty}(x_1^n, x_2^n) = (x_1,x_2)$. Let $\varepsilon>0$. There exist $N_1, N_2 \in \mathbb{N}$ such that $n\geq N_1$ implies $\|x_1^n - x_1\| < \frac{\varepsilon}{\sqrt{2}}$ and $n \geq N_2$ implies $\|x_2^n - x_2\| < \frac{\varepsilon}{\sqrt{2}}$. Thus, if $n \geq N := \max\{N_1, N_2\}$, then

$$\|(x_1^n, x_2^n) - (x_1,x_2)\| = (\|x_1^n - x_1\|^2 + \|x_2^n - x_2\|^2)^{\frac{1}{2}}$$
$$< \left(\left(\frac{\varepsilon}{\sqrt{2}}\right)^2 + \left(\frac{\varepsilon}{\sqrt{2}}\right)^2\right)^{\frac{1}{2}} = \varepsilon.$$

This finishes the proof.

b) $(\tilde{H}_1)^\perp = \{(x_1,x_2)\in H : \langle (x_1,x_2),(x_1',0)\rangle = 0$ for all $x_1'\in H_1\}$

$= \{(x_1,x_2)\in H : \langle x_1, x_1'\rangle = 0$ for all $x_1'\in H_1\}$

$= \{(0,x_2) : x_2\in H_2\} = \hat{H}_2.$

The argument for $(\tilde{H}_2)^\perp = \hat{H}_1$ is analogous.

We deduce that $H = \tilde{H}_1 \oplus (\tilde{H}_1)^\perp = \tilde{H}_1 \oplus \tilde{H}_2$.

4.8 a) We show that the inner product is well-defined: Suppose $f_1, f_2, g_1, g_2 \in \mathcal{L}^2(X,\mathcal{A},\mu)$ with $f_1 = f_2$ μ-a.e. and $g_1 = g_2$ μ-a.e. Then $\overline{g_1} = \overline{g_2}$ μ-a.e. Note also that if $x\in X$ is such that $f_1(x)\overline{g_1(x)} \neq f_2(x)\overline{g_2(x)}$ then either $f_1(x)\neq f_2(x)$ or $\overline{g_1(x)} \neq \overline{g_2(x)}$, and so

$$\{x\in X : f_1(x)\overline{g_1(x)} \neq f_2(x)\overline{g_2(x)}\}$$
$$\subseteq \{x\in X : f_1(x)\neq f_2(x)\} \cup \{x\in X : \overline{g_1(x)} \neq \overline{g_2(x)}\}$$

which gives that the measure of the former is zero.

Hence

$$\int f_1\overline{g_1}\,d\mu = \int f_2\overline{g_2}\,d\mu$$

which shows that the inner product is well-defined.

Linearity in the first argument goes as follows:

$$\int_X (\lambda f_1 + \mu f_2)\overline{g}\,d\mu = \lambda\int_X f_1\overline{g}\,d\mu + \mu\int_X f_2\overline{g}\,d\mu$$ [Just linearity of the integral]

conjugate linearity: $\overline{\langle f,g\rangle} = \overline{\int_X f\overline{g}\,d\mu} = \int_X \overline{f\overline{g}}\,d\mu = \int_X \overline{f}g\,d\mu$

$$= \int_X g\overline{f}\,d\mu = \langle g,f\rangle$$

Finally, $\langle f,f\rangle = \int_X f\bar{f}\,d\mu = \int_X |f|^2 d\mu \geq 0$ and

$\int_X |f|^2 d\mu = 0$ implies that $|f|^2 = 0$ μ-a.e., hence

$f=0$ μ-a.e.

b) Define a map $\phi : M_E \to L^2(E, \mathcal{A}_E, \mu_E)$

by $\phi(f) = f|_E$. To see that this is well-defined,

let $f \in M_E$, so that $f : X \to \mathbb{C}$ is measurable

and $\mu(\{x \in X \setminus E : f(x) \neq 0\}) = 0$. Then for $O \subseteq \mathbb{C}$ open

$$(f|_E)^{-1}(O) = \{x \in E : f(x) \in O\}$$
$$= E \cap f^{-1}(O)$$

This shows that $f|_E$ is $\mathcal{A}_E$-measurable.

Moreover,

$$\|f\|^2_{L^2(X,\mathcal{A},\mu)} = \int_X |f|^2 d\mu = \underbrace{\int_{X\setminus E} |f|^2 d\mu}_{\substack{=0 \text{ since } f=0 \\ \mu\text{-a.e. on } X\setminus E}} + \int_E |f|^2 d\mu = \int_E |f|^2 d\mu$$

$= \|f|_E\|^2_{L^2(E,\mathcal{A}_E,\mu_E)}$. This shows that ϕ is an isometry, and so it is injective.

It is straightforward to show that ϕ is linear.

To show surjectiveness, suppose $g \in L^2(E, \mathcal{A}_E, \mu_E)$.

Then let $f : X \to \mathbb{C}$ be given by

$$f(x) = \begin{cases} g(x) ; & x \in E \\ 0 ; & x \notin E \end{cases} = g \mathbb{1}_E$$

Then one has $f \in \mathcal{L}^2(X, \mathcal{A}, \mu)$ and $f|_E = g$.

4.9. <u>a) ⟹ b)</u>: Let $C = \{x_n : n \in \mathbb{N}\}$ be a countable, dense subset of H. Then

$$H \supseteq \overline{\operatorname{span}\{x_n : n \in \mathbb{N}\}} \supseteq \overline{\{x_n : n \in \mathbb{N}\}} = H$$

which shows that $(x_n)_{n\in\mathbb{N}}$ satisfies the property in Example 4.2.5.

<u>b) ⟹ c)</u>: This is Example 4.2.5.

c) ⟹ a): Suppose $\mathcal{B} = \{x_n : n \in \mathbb{N}\}$ is a countable orthonormal basis for H. Let $\mathbb{Q}(i) = \{a+bi : a, b \in \mathbb{Q}\} \subseteq \mathbb{C}$. Since $\mathbb{Q}$ is countable, $\mathbb{Q}(i)$ is countable, and since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\mathbb{Q}(i)$ is dense in $\mathbb{C}$. Let

$$S_n = \{\lambda_1 x_1 + \dots + \lambda_n x_n : \lambda_1, \dots, \lambda_n \in \mathbb{Q}(i)\}; \quad n \in \mathbb{N}.$$

The map $\beta : \mathbb{Q}(i)^n \to S^n$, $\beta(\lambda_1, \dots, \lambda_n) = \sum_{j=1}^{n} \lambda_j x_j$ is a surjection. Since $\mathbb{Q}(i)$ is countable, $\mathbb{Q}(i)^n$ is also countable, and so S^n must be countable. But then the countable union

$$S = \bigcup_{n=1}^{\infty} S_n \quad \text{is countable.}$$

We show that S is dense in H. Let $x \in H$, and $\varepsilon > 0$. Write $x = \sum_{n=1}^{\infty} \mu_n x_n$ for $\mu_n \in \mathbb{C}$. Since the sum is finite, there exists $N \in \mathbb{N}$ such that $\sum_{n>N} |\mu_n|^2 \underset{\text{Pythagoras}}{=} \|\sum_{n>N} \mu_n x_n\|^2 < \frac{\varepsilon^2}{4}$.

Pick $\lambda_n \in \mathbb{Q}(i)$ with $|\mu_n - \lambda_n| < \frac{\varepsilon}{2N^{1/2}}$ for each $n = 1, \dots, N$. Then

$$\|x - \sum_{n=1}^{N} \lambda_n x_n\| \le \|\sum_{n=1}^{N} (\mu_n - \lambda_n) x_n\| + \|\sum_{n>N} \lambda_n x_n\|$$

$$\leq \left(\sum_{n=1}^{N} |\mu_n - \lambda_n|^2 \right)^{\frac{1}{2}} + \left(\sum_{n>N} |\lambda_n|^2 \right)^{\frac{1}{2}}$$

$$< \left(\sum_{n=1}^{N} \left(\frac{\varepsilon}{2N^{1/2}} \right)^2 \right)^{\frac{1}{2}} + \left(\frac{\varepsilon^2}{4} \right)^{\frac{1}{2}}$$

$$= \left(N \cdot \frac{\varepsilon^2}{4N} \right)^{\frac{1}{2}} + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that S is dense in H since $\sum_{n=1}^{N} \lambda_n x_n \in S$.

4.10 Let $\mathcal{B} = \mathcal{B}_1 \times \{0\} \cup \{0\} \times \mathcal{B}_2$. We show that $\mathcal{B}$ is an orthonormal basis for H. We consider three exhaustive cases:

1) If $(b_1, 0), (b_2, 0) \in \mathcal{B}_1 \times \{0\}$, then

$$\langle (b_1, 0), (b_2, 0) \rangle = \langle b_1, b_2 \rangle + \langle 0, 0 \rangle = \langle b_1, b_2 \rangle$$

$$= \begin{cases} 1 ; & b_1 = b_2 \\ 0 ; & \text{otherwise} \end{cases}$$

Since $(b_1, 0) = (b_2, 0)$ iff $b_1 = b_2$, we're ok.

2) If $(b_1, 0) \in \mathcal{B}_1 \times \{0\}$, $(0, b_2) \in \{0\} \times \mathcal{B}_2$, then

$$\langle (b_1, 0), (0, b_2) \rangle = \langle b_1, 0 \rangle + \langle 0, b_2 \rangle = 0.$$

Since $(b_1, 0) \neq (0, b_2)$ (otherwise we would have $0 \in \mathcal{B}_1$ which is impossible)

we're ok.

3) $(0, b_1), (0, b_2) \in \{0\} \times \mathcal{B}_2$. Similar to 1).

This shows that $\mathcal{B}$ is orthonormal.

Now suppose $(x_1, x_2) \in \mathcal{B}^{\perp}$. Then for all $b_1 \in \mathcal{B}_1$ and $b_2 \in \mathcal{B}_2$ we have

$$0 = \langle (x_1, x_2), (b_1, 0) \rangle = \langle x_1, b_1 \rangle + \langle x_2, 0 \rangle = \langle x_1, b_1 \rangle$$

which gives $x_1 \in \mathcal{B}_1^{\perp} \rightsquigarrow x_1 = 0$, and

$$0 = \langle (x_1, x_2), (0, b_2) \rangle = \langle x_1, 0 \rangle + \langle x_2, b_2 \rangle = \langle x_2, b_2 \rangle$$

which gives $x_2 \in \mathcal{B}_2^{\perp} \rightsquigarrow x_2 = 0$.

Hence $(x_1, x_2) = (0, 0)$, and so

$$\mathcal{B}^{\perp} = \{(0,0)\}.$$

This proves that $\mathcal{B}$ is an orthonormal basis for $\mathcal{H}$.