MAT 3400/4400 Spring 2019 April 23

SOLUTIONS

<u>4.12</u> We first prove the following: If J is countable and $(x_j)_{j\in J} \subseteq X$ a normed space with $\sum_{j\in J} x_j = x$ in the sense of Theorem 4.2.8., i.e. for any enumeration $\{j_k : k\in\mathbb{N}\}$ of J, we have $\sum_{k=1}^{\infty} x_{j_k} = x$, then the generalized sum $\sum_{j\in J} x_j$ converges to x.

So suppose for every enumeration $\{j_k : k\in\mathbb{N}\}$, of J, $\sum_{k=1}^{\infty} x_{n_j} = x$. Suppose for a contradiction that the generalized sum $\sum_{j\in J} x_j$ does not converge to x. Then

$$\exists\, \varepsilon>0 \quad \forall\, F_0 \overset{\text{finite}}{\subseteq} J \quad \exists\, F \overset{\text{finite}}{\supseteq} F_0 : \quad \Big\| \sum_{j\in F} x_j - x \Big\| \geq \varepsilon. \quad (*)$$

Let $\{j_n : n\in\mathbb{N}\}$ be some enumeration of J. Then by assumption $\exists\, M_1>0$ with $\forall\, N\geq M_1$:
$\Big\| \sum_{n=1}^{N} x_{j_n} - x \Big\| < \varepsilon/2$. Define $F_1 = \{j_1,\dots,j_{M_1}\}$. Then by $(*)$, $\exists\, G_1 \overset{\text{finite}}{\supseteq} F_1$ with $\Big\| \sum_{j\in G_1} x_j - x \Big\| \geq \varepsilon$. Let $M_2 = \max\{n : j_n \in G_1\}$, and let $F_2 = \{j_1,\dots,j_{M_2}\}$.

Continuing, we obtain a sequence of $\overset{\text{finite}}{\text{sets}}$ $F_1 \subseteq G_1 \subseteq F_2 \subseteq G_2 \subseteq \dots$ with

$$\Big\| \sum_{j\in F_N} x_j - x \Big\| < \frac{\varepsilon}{2} \quad \text{and} \quad \Big\| \sum_{j\in G_N} x_j - x \Big\| \geq \varepsilon.$$

Hence $\Big\| \sum_{j\in G_N\setminus F_N} x_j \Big\| = \Big\| \sum_{j\in G_N} x_j - \sum_{j\in F_N} x_j \Big\| \geq$

$$\Big\| \sum_{j\in G_N} x_j - x \Big\| - \Big\| x - \sum_{j\in F_N} x_j \Big\| \geq \varepsilon - \frac{\varepsilon}{2} = \varepsilon.$$

This shows that F_N is a proper subset of G_N for each N. Let $\{r_n : n \in \mathbb{N}\}$ be a new enumeration of the set J where F_1 is enumerated first, then $G_1 \setminus F_1$, then $F_2 \setminus G_1$, etc. For each N we have

$$\left\| \sum_{n=|F_N|+1}^{|G_N|} x_{j_n} \right\| = \left\| \sum_{n \in G_N \setminus F_N} x_{j_n} \right\| \geq \frac{\varepsilon}{2}$$

Since $|F_N|$ and $|G_N|$ go to infinity as N increases, this shows that $\left(\sum_{n=1}^{N} x_{j_n} \right)_N$ is not Cauchy, and so we have a contradiction. Hence the generalized sum $\sum_{j \in J} x_j$ converges to x.

Now let H be a Hilbert space, $x \in H$, $\mathcal{B}$ an orthonormal basis. Then we have

$$\sum_{u \in \mathcal{B}} \langle x, u \rangle u = \sum_{u \in \mathcal{B}_x} \langle x, u \rangle u + \sum_{u \in \mathcal{B} \setminus \mathcal{B}_x} \langle x, u \rangle u$$

$$= \sum_{u \in \mathcal{B}_x} \langle x, u \rangle u$$

[Here we take for granted some basic properties of generalized sums]

By what we proved, since $\sum_{u \in \mathcal{B}_x} \langle x, u \rangle u$ converges in the sense of THM 4.2.8 to x, we get that the generalized sum $\sum_{u \in \mathcal{B}_x} \langle x, u \rangle u$ converges to x. By the above, we get that the generalized sum $\sum_{u \in \mathcal{B}} \langle x, u \rangle u$ converges to x. Exactly the same argument holds for the case $P_M(x) = \sum_{u \in \mathcal{E}} \langle x, u \rangle u$.

4.13 c) Let $g : [-\pi, \pi] \to \mathbb{C}$, $g(t) = e^t$.

$$\hat{g}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^t e^{-int}\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(1-in)t}\, dt$$

$$= \frac{1}{2\pi} \frac{1}{1-in} e^{(1-in)t} \Big|_{-\pi}^{\pi} = \frac{1}{2\pi(1-in)} \left(e^{(1-in)\pi} - e^{(1-in)(-\pi)} \right)$$

$$= \frac{1+in}{2\pi(1-in)(1+in)} \left(e^{\pi} e^{-in\pi} - e^{-\pi} e^{in\pi} \right)$$

$$= \frac{1+in}{2\pi(1+n^2)} \left(e^{\pi}(-1)^n - e^{-\pi}(-1)^n \right)$$

$$= \frac{(-1)^n}{\pi} \frac{1+in}{1+n^2} \quad (\pi)$$

Thus:

$$|\hat{g}(n)|^2 = \frac{1}{\pi^2} \frac{1+n^2}{(1+n^2)} (\sinh(\pi))^2 = \frac{\sinh(\pi)^2}{\pi^2} \frac{1}{1+n^2}$$

Now

$$\|g\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(t)|^2\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{2t}\, dt = \frac{1}{4\pi} \left(e^{2\pi} - e^{-2\pi} \right)$$

$$= \frac{1}{2\pi} \sinh(\pi).$$

By Parseval:

$$\frac{1}{2\pi} \sinh(2\pi) = \|g\|_2^2 = \sum_{n \in \mathbb{Z}} |\hat{g}(n)|^2 = \frac{\sinh(\pi)^2}{\pi^2} \sum_{n \in \mathbb{Z}} \frac{1}{1+n^2}$$

$$= \frac{\sinh(\pi)^2}{\pi^2} \left(1 + 2 \sum_{n=1}^{\infty} \frac{1}{1+n^2} \right)$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{1}{2} \left[\frac{1}{2\pi} \sinh(2\pi) \cdot \frac{\pi^2}{\sinh(\pi)^2} - 1 \right]$$

$$= \frac{1}{2} \left[\frac{\pi \sinh(2\pi)}{2 \sinh(\pi)^2} - 1 \right].$$

4.14 a) Let $T: H \to H$ be given by $Tf(t) = f(-t)$ for $f \in H$, $t \in [-\pi, \pi]$. Then T is easily shown to be well-defined and linear. Moreover,

$$\|Tf\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(-t)|^2 dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^2 dt = \|f\|_2^2$$

which shows that T is bounded. Hence $I - T$ is bounded and linear, and so its kernel is closed. But

$$\begin{aligned} \operatorname{Ker}(I-T) &= \{ f \in H : \forall t \in [-\pi,\pi] : (I-T)f(t) = 0 \} \\ &= \{ f \in H : \forall t \in [-\pi,\pi] : f(t) - f(-t) = 0 \} \\ &= H_{even}. \end{aligned}$$

This shows that H_{even} is closed.

Now suppose $f \in H_{even}$. Then for $g \in H_{odd}$, we have

$$(f,g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\overline{g(t)}\, dt = \frac{1}{2\pi} \left(\int_{-\pi}^{0} f(t)\overline{g(t)}\, dt + \int_{0}^{\pi} f(t)\overline{g(t)}\, dt \right)$$

$$= \frac{1}{2\pi} \left(\int_{0}^{\pi} f(-t)\overline{g(-t)}\, dt + \int_{0}^{\pi} f(t)\overline{g(t)}\, dt \right)$$

$$= \frac{1}{2\pi} \left(- \int_{0}^{\pi} f(t)\overline{g(t)}\, dt + \int_{0}^{\pi} f(t)\overline{g(t)}\, dt \right) = 0.$$

Hence $f \in (H_{odd})^{\perp}$. Conversely, let $f \in (H_{odd})^{\perp}$. Since

$$f(t) = \underbrace{\tfrac{1}{2}(f(t) + f(-t))}_{g(t)} + \underbrace{\tfrac{1}{2}(f(t) - f(-t))}_{h(t)} \qquad (*)$$

where g is even and h is odd, we have

1) $\langle f, f \rangle = \langle f, g+h \rangle = \langle f, g \rangle + \langle f, h \rangle = \langle f, g \rangle$
$= \langle g+h, g \rangle = \langle g, g \rangle + \langle h, g \rangle = \langle g, g \rangle.$

On the other hand,

$$\langle f,f\rangle = \langle g+h, g+h\rangle = \langle g,g\rangle + \langle g,h\rangle + \langle h,g\rangle + \langle h,h\rangle$$
$$= \langle g,g\rangle + \langle h,h\rangle.$$

This shows that

$$\langle g,g\rangle = \langle g,g\rangle + \langle h,h\rangle$$

and so $\langle h,h\rangle = 0$, i.e. $h=0$. Hence $f = g \in H_{even}$,

and so $\qquad H_{even} = (H_{odd})^{\perp}$.

We now have $\qquad H = H_{even} \oplus H_{odd}$.

If $f = g+h$ where g is even and h is odd, this means that g is the orthogonal projection of f onto H_{even}.
But from $(*)$, we know that $g(t) = \frac{1}{2}(f(t) + f(-t))$ is such a function. Hence the projection P is given by

$$Pf(t) = \frac{1}{2}(f(t) + f(-t)).$$

b) Let $(e_n)_{n\in\mathbb{Z}}$ denote the standard orthonormal basis for H, i.e. $e_n(t) = e^{int}$. Note that

$$Pe_n(t) = \frac{1}{2}(e^{int} + e^{-int}) = \cos(nt).$$

If $f \in H_{even}$, then $f = Pf = P\left(\sum_{n\in\mathbb{Z}} \langle f, e_n\rangle e_n\right)$

$$= \sum_{n\in\mathbb{Z}} \langle f, e_n\rangle Pe_n = \sum_{n\in\mathbb{Z}} \langle Pf, e_n\rangle Pe_n = \sum_{n\in\mathbb{Z}} \langle f, Pe_n\rangle Pe_n.$$

This shows that $\overline{\text{span}\{Pe_n\}} = H_{even}$. However, $\{Pe_n\}_{n\in\mathbb{Z}}$ is not orthonormal. Indeed, $Pe_n = Pe_{-n}$ for all $n \in \mathbb{Z}$.

But we claim that $\{Pe_n\}_{n=0}^{\infty}$ is orthogonal:

First of all, $\langle Pe_0, Pe_0\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\cos(0t)|^2\, dt$

$= 1.$

Secondly, if $m, n \geq 0$ and not both equal to 0, then

$\langle Pe_m, Pe_n\rangle = \langle \frac{1}{2}(e_m + e_{-m}), \frac{1}{2}(e_n + e_{-n})\rangle$

$= \frac{1}{4}\left(\langle e_m, e_n\rangle + \underbrace{\langle e_m, e_{-n}\rangle}_{0} + \underbrace{\langle e_{-m}, e_n\rangle}_{0} + \langle e_{-m}, e_{-n}\rangle\right)$

$= \frac{1}{4}\left(\delta_{m,n} + \delta_{-m,-n}\right) = \frac{\delta_{m,n}}{2}.$

Hence, with normalizations, $\{Pe_0, 2Pe_1, 2Pe_2, \ldots\}$

$= \{1, 2\cos(t), 2\cos(2t), \ldots\}$ is an orthonormal

basis for H_{even}.

A similar argument using sines gives an orthonormal basis for H_{odd}.

4.15 By Theorem 3.3.2, if $x \in H$ and $(x_n)_{n=1}^{\infty}$ is a sequence in H_0 that converges to x, then $Sx = \lim_{n\to\infty} S_0 x_n$. Now let $x, y \in H$ and let $(x_n)_n$, $(y_n)_n$ be sequences in H_0 that converge to x and y respectively. Then by continuity,

$$\langle T(x), y\rangle = \langle T(\lim_{n\to\infty} x_n), \lim_{n\to\infty} y_n\rangle$$

$$= \langle \lim_{n\to\infty} T_0(x_n), \lim_{n\to\infty} y_n\rangle$$

$$= \lim_{n\to\infty} \langle T_0(x_n), y_n\rangle$$

$$= \lim_{n\to\infty} \langle x_n, S_0(y_n)\rangle$$

$$= \langle \lim_{n\to\infty} x_n, \lim_{n\to\infty} S_0(y_n)\rangle$$

$$= \langle x, S(y)\rangle.$$

This shows that $T^* = S$.

4.16 Let $H_0 = \{[g] : g \in C([a,b])\} \subseteq H = L^2([a,b])$, which is a dense subspace by Example 4.3.7.

Let $f, g \in C([a,b])$. Then

$$\langle T_K f, g\rangle = \int_a^b T_K f(t)\overline{g(t)}\,dt = \int_a^b\int_a^b K(t,s) f(s)\,ds\,\overline{g(t)}\,dt$$

$$= \int_a^b\int_a^b f(s)\,\overline{\overline{K(t,s)}g(t)}\,dt\,ds = \int_a^b f(s)\,\overline{T_{K^*}g(s)}\,ds$$

$$= \langle f, T_{K^*}g\rangle.$$ By exercise 4.15, we obtain the desired conclusion.

4.18 a) Let $x \in H$. Then

$$\|T_{v,w}x\| = \|\langle x,v\rangle w\| \leq |\langle x,v\rangle| \|w\| \leq \|x\| \|v\| \|w\|.$$

This shows that $T_{v,w}$ is bounded, with $\|T_{v,w}\| \leq \|v\| \|w\|$.

Furthermore, $\|T_{v,w}(\frac{v}{\|v\|})\| = \|\langle \frac{v}{\|v\|}, v\rangle w\|$

$= \|\frac{1}{\|v\|}\|v\|^2 w\| = \|v\| \|w\|$, which shows that

$\|T_{v,w}\| = \|v\| \|w\|$. Finally, for $x,y \in H$, we have

$$\langle T_{v,w}x, y\rangle = \langle \langle x,v\rangle w, y\rangle = \langle x,v\rangle \langle w,y\rangle$$

$$= \langle x, \overline{\langle w,y\rangle} v\rangle = \langle x, \langle y,w\rangle v\rangle = \langle x, T_{w,v}y\rangle$$

which shows that $T_{v,w}^* = T_{w,v}$.

b) Suppose $T \in B(H)$ has rank 1. Then $T(H)$ is 1-dimensional, say $T(H) = \operatorname{span}\{w\}$ for some $w \in H$, $w \neq 0$. Then for every $x \in H$, there exists a unique scalar λ_x such that $T(x) = \lambda_x w$. Now

$$\lambda_{x+y} w = T(x+y) = T(x) + T(y) = \lambda_x w + \lambda_y w = (\lambda_x + \lambda_y) w$$

shows that the map $H \to \mathbb{C}$, $x \mapsto \lambda_x$, is linear.

Furthermore,

$$|\lambda_x| \|w\| = \|\lambda_x w\| = \|T(x)\| \leq \|T\| \|x\|$$

$\Rightarrow |\lambda_x| \leq \frac{\|T\|}{\|w\|} \|x\|$, and so λ_x defines a bounded linear functional. By Riesz' representation theorem (Theorem 4.3.1), there exists a unique $v \in H$ such that $\lambda_x = \langle x, v\rangle$. Hence $T(x) = \langle x,v\rangle w$, and so

$$T = T_{v,w}.$$

c) Let T have finite rank, say n. Let $\{w_1, \ldots, w_n\}$ be an orthonormal basis for $T(H)$. Then for each $x \in H$, there exist unique scalars $\lambda_x^{(1)}, \ldots, \lambda_x^{(n)} \in \mathbb{C}$ such that

$$T(x) = \sum_{k=1}^{n} \lambda_x^{(k)} w_k .$$

As before, the mappings $x \mapsto \lambda_x^{(k)}$ for each k are linear. Furthermore, for each k,

$$|\lambda_x^{(k)}|^2 = \|\lambda_x^{(k)} w_k\|^2 \leq \sum_{k'=1}^{n} \|\lambda_x^{(k')} w_{k'}\|^2$$

$$= \left\| \sum_{k'=1}^{n} \lambda_x^{(k')} w_{k'} \right\|^2 = \|T(x)\|^2 \leq \|T\|^2 \|x\|^2$$

and so $|\lambda_x^{(k)}| \leq \|T\| \|x\|$, which shows that $x \mapsto \lambda_x^{(k)}$ are bounded linear functionals. By Riesz' as before, we get that $\lambda_x^{(k)} = \langle x, v_k \rangle$ for vectors v_k, and so $T = \sum_{k=1}^{n} T_{v_k, w_k}$.

d) If T is finite rank, then $T = \sum_{k=1}^{n} T_{v_k, w_k}$ for some $v_k, w_k \in H$, $k = 1, \ldots, n$, and so

$$T^* = \sum_{k=1}^{n} (T_{v_k, w_k})^* = \sum_{k=1}^{n} T_{w_k, v_k} .$$