MAT3400/4400 Exercise solutions April 29 (Spring 2019)

4.19 Let $T \in \mathcal{B}(H)$, $M \subseteq H$ closed subspace. Then

M is invariant under T

$\Updownarrow$

$\forall x \in M: \; Tx \in M$

$\Updownarrow$ (by $(M^\perp)^\perp = M$)

$\forall x \in M \; \forall y \in M^\perp: \; \langle Tx, y \rangle = 0$

$\Updownarrow$

$\forall y \in M^\perp \; \forall x \in M: \; \langle T^* y, x \rangle = 0$

$\Updownarrow$

$\forall y \in M^\perp: \; T^* y \in M^\perp$

$\Updownarrow$

$M^\perp$ is invariant under T^*.

4.20

a) $\operatorname{Ker}(T) = \operatorname{Ker}(T^*T)$:

If $Tx = 0$ then $T^*(Tx) = 0$ which shows that $\operatorname{Ker} T \subseteq \operatorname{Ker}(T^*T)$.

Conversely, if $T^*Tx = 0$ then

$0 = \langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2$

which implies $Tx = 0$, thus $\operatorname{Ker}(T^*T) \subseteq \operatorname{Ker} T$.

$\overline{T^*(H)} = \overline{T^*T(H)}$:

By Prop 4.3.8. we have that

By what we proved already

$$\overline{T^*(H)} = \operatorname{Ker}((T^*)^*)^\perp = \operatorname{Ker}(T)^\perp \overset{\downarrow}{=} \operatorname{Ker}(T^*T)^\perp$$

$$= \operatorname{Ker}((T^*T)^*)^\perp = \overline{T^*T(H)}.$$

b) If $T^*T = TT^*$, then by a),

$$\operatorname{Ker}(T) = \operatorname{Ker}(T^*T) = \operatorname{Ker}(TT^*) = \operatorname{Ker}((T^*)^*T)$$
$$= \operatorname{Ker}(T)$$

$$\overline{T^*(H)} = \overline{T^*T(H)} = \overline{TT^*(H)} = \overline{(T^*)^*T^*(H)}$$
$$= \overline{(T^*)^*(H)} = \overline{T(H)}.$$

c) If T is normal, then for all $\mu \in \mathbb{C}$,

$$(T-\mu I)^*(T-\mu I) = T^*T - \mu T^* - \bar{\mu} T + |\mu|^2 I$$
$$= TT^* - \bar{\mu} T - \mu T^* + |\mu|^2 I = (T-\mu I)(T-\mu I)^*,$$

showing that $T - \mu I$ is normal as well.

By b), we get that

$$\operatorname{Ker}(T-\mu I) = \operatorname{Ker}((T-\mu I)^*) = \operatorname{Ker}(T^* - \bar{\mu} I).$$

Hence, x is an eigenvector for T with eigenvalue λ if and only if x is an eigenvector for T^* with eigenvalue $\bar{\lambda}$. Moreover, since

$$E_\lambda^T = \operatorname{Ker}(T-\lambda I) \quad \text{and} \quad E_{\bar{\lambda}}^{T^*} = \operatorname{Ker}(T^* - \bar{\lambda} I)$$

we get that $E_\lambda^T = E_{\bar{\lambda}}^{T^*}$.

d) $Su_j = u_{j+1}$. $Tu_{j+1} = u_j$ and $Tu_1 = 0$.

Then $TT^* u_j = TSu_j = Tu_{j+1} = u_j$ for all $j \in \mathbb{N}$,

i.e. $TT^* = I_H$, but

$$T^*Tu_j = STu_j = \begin{cases} 0 ; & \text{if } j = 1 \\ u_j : & \text{if } j > 1 \end{cases}$$

i.e. T^*T is the orthogonal projection onto $\operatorname{span}\{u_2, u_3 \ldots\} = \{u_1\}^\perp$.

Since $Tu_1 = 0 = 0u_1$, we have that 0 is an eigenvalue for T. But if $Sx = 0$ for, say, $x = \sum_{j=1}^\infty c_j u_j \in H$, then $0 = Sx = \sum_{j=1}^\infty c_j Su_j$

$= \sum_{j=1}^\infty c_j u_{j+1} = \sum_{j=2}^\infty c_{j-1} u_j$, and so $c_{j-1} = 0$ for all $j = 2, 3, \ldots$, i.e. $x = 0$. Thus S does not have 0 as an eigenvalue.

S has no eigenvalues:

Suppose $Sx = \lambda x$ for $x \in H$, $x \neq 0$, $\lambda \in \mathbb{C}$, $\lambda \neq 0$.

Let $x = \sum_{j=1}^{\infty} c_j u_j$. Then we get

$$Sx = \sum_{j=2}^{\infty} c_{j-1} u_j = \sum_{j=1}^{\infty} (\lambda c_j) u_j, \quad \text{i.e.}$$

$$\lambda c_1 = 0$$
$$\lambda c_j = c_{j-1} \quad \text{for } j \geq 2.$$

Since $\lambda \neq 0$ we get $c_1 = 0$.

But then $c_j = c_{j-1}/\lambda$ for all $j \geq 2$ implies $c_j = 0$ for all $j \in \mathbb{N}$, so $x = 0$, a contradiction.

Hence S has no eigenvalues.

Suppose $\lambda \in \mathbb{C}$ with $|\lambda| < 1$, $\lambda \neq 0$. For λ to be an eigenvalue of T, we need $Tx = \lambda x$ for some $x \neq 0$ in H, say $x = \sum_{j=1}^{\infty} c_j u_j$. Then $Tx = \sum_{j=1}^{\infty} c_{j+1} u_j = \sum_{j=1}^{\infty} \lambda c_j u_j$

which implies $c_{j+1} = \frac{c_j}{\lambda}$ for all $j \geq 1$, i.e.

$c_j = \frac{c_1}{\lambda^{j-1}}$ for all $j \geq 1$. Now since $|\lambda| < 1$,

$$\|x\|^2 = \sum_{j=1}^{\infty} |c_j|^2 = |c_1|^2 \sum_{j=1}^{\infty} \frac{1}{|\lambda^2|^{j-1}} = |c_1|^2 \frac{1}{1-|\lambda|^2}$$

which shows that for any $c_1 \in \mathbb{C}$, x is an element of H with this choice of coefficients. Furthermore, we get $Tx = \lambda x$.

4.22 a) Suppose $(X, \mathcal{A}, \mu)$ is σ-finite.

Let $(X_k)_{k=1}^{\infty}$ be a sequence of measurable sets such that $\mu(X_k) < \infty$ for each k and

$$X = \bigcup_{k=1}^{\infty} X_k .$$

Let $E \in \mathcal{A}$ with $\mu(E) = \infty$. Then

$$E = \bigcup_{k=1}^{\infty} X_k \cap E$$

which implies $\infty = \mu(E) \leq \sum_{k=1}^{\infty} \mu(X_k \cap E)$.

Thus, there has to be a $k \in \mathbb{N}$ for which $\mu(X_k \cap E) > 0$. But since $X_k \cap E \subseteq X_k$, we get $\mu(X_k \cap E) \leq \mu(X_k) < \infty$. Also, $X_k \cap E \subseteq E$, so we have found our set $F \in \mathcal{A}$ with $F \subseteq E$ and $0 < \mu(F) < \infty$, namely $F = X_k \cap E$.

b) Let $g \in L^2(X, \mathcal{A}, \mu)$. Then

$$\| M_f g \|_2^2 = \int_X |fg|^2 \, d\mu \leq \int_X \|f\|_\infty |g|^2 \, d\mu = \|f\|_\infty \|g\|_2$$

which shows that $\|M_f\| \leq \|f\|_\infty$.

Now let $\varepsilon > 0$, and consider the set

$A = \{ x \in X : |f(x)| \geq \|f\|_\infty - \varepsilon \}$.

If $\mu(A) = 0$ then $|f| < \|f\|_\infty - \varepsilon$ μ-a.e. which implies $\|f\|_\infty \leq \|f\|_\infty - \varepsilon$, a contradiction. Hence, $\mu(A) > 0$. If $\mu(A) = \infty$, we can find $B \in \mathcal{A}$, $B \subseteq A$ with $0 < \mu(B) < \infty$. If $\mu(A) < \infty$, just set $B = A$.

Now

$$\|M_f 1_B\|_2^2 = \int_X |f 1_B|^2 d\mu = \int_B |f|^2 d\mu$$

$$\geq \mu(B)\,(\|f\|_\infty - \varepsilon)^2$$ and so

$$\|M_f 1_B\|_2 \geq \mu(B)^{1/2}(\|f\|_\infty - \varepsilon).$$

Hence, with $g = \mu(B)^{-1/2} 1_B$, we get $\|g\|_2 = 1$, and $\|M_f g\|_2 \geq \|f\|_\infty - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this shows that $\|M_f\| = \sup_{\|h\|_2 = 1} \|M_f h\|_2 \geq \|f\|_\infty$, and so we have $\|M_f\| = \|f\|_\infty$.

c) Suppose M_f is self-adjoint. By Example 4.3.6., $M_f^* = M_{\bar{f}}$, so we get $M_{\bar{f}} = M_f$, i.e.

$$f g = \bar{f} g \quad \mu\text{-a.e.} \qquad (*)$$

for all $g \in L^2(X, \mathcal{A}, \mu)$. Let $N = \{x \in X : f(x) \neq \overline{f(x)}\}$.
Suppose for a contradiction that $\mu(N) > 0$.
If $\mu(N) = \infty$, then we can find $N' \in \mathcal{A}$, $N' \subseteq N$ with $0 < \mu(N') < \infty$ (by semi finiteness). Otherwise chose $N' = N$. Since $\mu(N') < \infty$, $(*)$ gives that $f 1_{N'} = \bar{f} 1_{N'}$ μ-a.e., i.e. $N'' = \{x \in N' : f(x) \neq \overline{f(x)}\}$ has $\mu(N'') = 0$. But if $x \in N'$ then $f(x) \neq \overline{f(x)}$ since ~~$N \subseteq N'$~~ $N' \subseteq N$, hence $N'' = N'$. It follows that $\mu(N'')=\mu(N')$, a contradiction. Hence $\mu(N) > 0$, and so $f(x) = \overline{f(x)}$ μ-a.e., i.e. f is real-valued μ-a.e.

4.24 a) $N_T = \sup \{ |\langle T(x), x \rangle| : x \in H, \|x\| = 1 \}$.

By Theorem 4.4.9., $\|T\| = N_T$ when $T \in \mathcal{B}(H)$ is self-adjoint. Consequently, if $\langle T(x), x \rangle = 0$ for all $x \in H$ and T is self-adjoint, then $\|T\| = 0$, forcing $T = 0$. Conversely, if $T = 0$, then $\langle T(x), x \rangle = 0$ for all $x \in H$.

b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by 90 degrees, i.e. $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then $\langle Tx, x \rangle = 0$ for all $x \in \mathbb{R}^2$ by the geometrical interpretation of the inner product. However, $T \neq 0$.

c) By Corollary 4.4.8., we write

$T = Re(T) + i\, Im(T)$ with $Re(T), Im(T)$ both self-adjoint. Suppose $\langle T(x), x \rangle = 0$ for all $x \in H$.

Then

$$(\Delta) \qquad 0 = \langle T(x), x \rangle = \langle Re(T)(x), x \rangle + i \langle Im(T)(x), x \rangle$$

for all $x \in H$. Since $N_S \subseteq \mathbb{R}$ for self-adjoint operators $S \in \mathcal{B}(H)$, we have that

$$\langle Re(T)(x), x \rangle, \langle Im(T)(x), x \rangle \in \mathbb{R}$$

for all $x \in H$. Hence by (Δ), we obtain

$$\langle Re(T)(x), x \rangle = 0 = \langle Im(T)(x), x \rangle$$

for all $x \in H$. But then by a), $Re(T) = Im(T) = 0$, and so $T = Re(T) + i\, Im(T) = 0$.

4.26 (i) $\Rightarrow$ (ii): Already seen.

(ii) $\Rightarrow$ (iii): Suppose $W_T \subseteq \mathbb{R}$, i.e.

$\langle T(x), x\rangle \in \mathbb{R}$ for all $x \in H$ with $\|x\| = 1$.

Let $x \in H$. Then $\frac{x}{\|x\|}$ has norm 1, and so by

assumption

$$\mathbb{R} \ni \left\langle T\left(\frac{x}{\|x\|}\right), \frac{x}{\|x\|}\right\rangle = \frac{1}{\|x\|^2}\langle T(x), x\rangle.$$

Thus, since $\frac{1}{\|x\|^2} \geq 0$, we have $\langle T(x), x\rangle \in \mathbb{R}$.

(iii) $\Rightarrow$ (i): Suppose $\langle T(x), x\rangle \in \mathbb{R}$ for all $x \in H$.

Then for any $x \in H$:

$$\langle T^*(x), x\rangle = \langle x, T(x)\rangle = \overline{\langle T(x), x\rangle}$$

$$\overset{\text{By assumption}}{=} \langle T(x), x\rangle.$$

Hence $\langle (T^* - T)(x), x\rangle = \langle T^*(x), x\rangle - \langle T(x), x\rangle$
$= 0$

for all $x \in H$, and so $T = T^*$ by Exercise 4.24 c).

4.27 a) Since

$$\langle S^*Sx, x\rangle = \langle Sx, Sx\rangle = \|Sx\|^2 \geq 0,$$

it follows that S^*S is positive. Moreover,

$$\langle R^2x, x\rangle \underset{\substack{\uparrow \\ R \text{ is self-adjoint}}}{=} \langle Rx, Rx\rangle = \|Rx\|^2 \geq 0$$

so R^2 is positive as well. Now

$$\|S\| \leq 1 \iff \forall x \in H: \|S(x)\|^2 \leq \|x\|^2$$

$$\iff \forall x \in H: \|x\|^2 - \|S(x)\|^2 \geq 0.$$

Finally, notice that

$$\langle (I - S^*S)x, x\rangle = \langle x, x\rangle - \langle S^*Sx, x\rangle$$

$$= \|x\|^2 - \|S(x)\|^2$$

and so $I - S^*S$ is positive if and only if $\|S\| \leq 1$.

b) Since $P_M = P_M^2 = P_M^*$, we have that

$P_M = P_M^* P_M$, and so by a), P_M is positive.

c) If T, T' are positive, then

$$\langle (T+T')x, x\rangle = \underbrace{\langle T(x), x\rangle}_{\geq 0} + \underbrace{\langle T'(x), x\rangle}_{\geq 0} \geq 0$$

showing that $T+T'$ is positive. Moreover, $\langle \lambda Tx, x\rangle = \underbrace{\lambda}_{\geq 0} \underbrace{\langle T(x), x\rangle}_{\geq 0} \geq 0$,

showing that λT is positive.

4.28 Suppose λ is an eigenvalue of M_f, with eigenvector $g \in H$, $g \neq 0$. Then

$$M_f(g) = fg = \lambda g \quad \mu\text{-a.e. on } [0,1]$$

which implies

$$fg = \lambda g \quad \mu\text{-a.e. on } [0,1] \setminus \{\lambda\},$$

since $\mu(\{\lambda\}) = 0$.

But
$$\begin{aligned} &\{x \in [0,1] \setminus \{\lambda\} : fg(x) \neq \lambda g(x)\} \quad (*) \\ &= \{x \in [0,1] \setminus \{\lambda\} : (x-\lambda) g(x) \neq 0\} \\ &= \{x \in [0,1] \setminus \{\lambda\} : g(x) \neq 0\}. \quad (**) \end{aligned}$$

Since $(*)$ has measure zero as shown, it follows that $(**)$ has measure zero, and so $g = 0$ μ-a.e. on $[0,1] \setminus \{\lambda\}$, from which it follows that $g = 0$ μ-a.e. on $[0,1]$. This is a contradiction, hence M_f has no eigenvalues.