MAT3400/4400 Spring 2019 Exercises

May 6th, SOLUTIONS

Exercise 4.29 Suppose M_f is unitary, where $f \in \mathcal{L}^\infty$. Since $M_f^* = M_{\bar f}$, we get that

$$I = M_f^* M_f = M_{\bar f} M_f = M_{\bar f f} = M_{|f|^2}$$

, i.e. that $|f|^2 g = g$ holds pointwise μ-a.e. for all $g \in \mathcal{L}^2$. Let $N = \{ x \in X : |f(x)| \neq 1 \}$. Suppose for a contradiction that $\mu(N) > 0$. If $\mu(N) = \infty$, then by semifiniteness, there exists $N' \in \mathcal{A}$, $N' \subseteq N$ with $0 < \mu(N') < \infty$. Otherwise let $N' = N$. Now $g = 1_{N'} \in \mathcal{L}^2$ since $\mu(N') < \infty$ so we get that $|f|^2 1_{N'} = 1_{N'}$ pointwise μ-a.e. But then $\mu(M) = 0$, where

$$M = \{ x \in N' : |f(x)| \neq 1 \}.$$

Since $N' \subseteq N$, we have that $M = N'$. But then both $\mu(N) = 0$ and $\mu(N') > 0$, a contradiction. We conclude that $\mu(N) = 0$, so that $|f| = 1$ pointwise μ-a.e.

Exercise 4.30

a) Let $x, y \in H$. Then

$$\langle V(x), V(y)\rangle = \left\langle \sum_{k\in\mathbb{Z}} \langle x, v_k\rangle v_{k+1}, \sum_{l\in\mathbb{Z}} \langle y, v_l\rangle v_{l+1}\right\rangle$$

$$= \sum_{k,l\in\mathbb{Z}} \langle x, v_k\rangle \langle y, v_l\rangle \langle v_{k+1}, v_{l+1}\rangle = \sum_{k,l\in\mathbb{Z}} \langle x, v_k\rangle \overline{\langle y, v_l\rangle}\, \delta_{k,l}$$

$$= \sum_{k\in\mathbb{Z}} \langle x, v_k\rangle \overline{\langle y, v_k\rangle} = \left\langle x, \sum_{k\in\mathbb{Z}} \langle y, v_k\rangle v_k\right\rangle = \langle x, y\rangle.$$

This shows that V is an isometry. However, if $k \in \mathbb{Z}$ then

$$V(v_{k-1}) = \sum_{l\in\mathbb{Z}} \langle v_{k-1}, v_l\rangle v_{l+1} = v_{(k-1)+1} = v_k.$$

This shows that every v_k is in the range of V. Thus V is surjective, since $\{v_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis for H. Linearity of V is easy to show. By Proposition 4.5.5., V is a unitary operator.

b) Let $f \in H = L^2([-\pi,\pi])$. Then

$$V(f) = \sum_{k\in\mathbb{Z}} \langle f, v_k\rangle v_{k+1}.$$ In particular, for every $l \in \mathbb{Z}$:

$$V(v_l) = \sum_{k\in\mathbb{Z}} \langle v_l, v_k\rangle v_{k+1} = \sum_{k\in\mathbb{Z}} \delta_{k,l} v_{k+1} = v_{l+1}.$$

Note that if $e(t) = e^{int}$, then for $t \in [-\pi,\pi]$, we get

$$M_e(v_l)(t) = e \cdot v_l(t) = e^{it} e^{ilt} = e^{i(l+1)t} = v_{l+1}(t).$$

Thus $M_e(v_l) = v_{l+1}$. Since M_e and V agree on an orthonormal basis, we conclude that $M_e = V$.

Exercise 5.1 a) Let $T, T' \in \mathbb{K}(X,Y)$ and $\lambda \in \mathbb{C}$.

Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence in X. Then by compactness of T, there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $(T(x_{n_k}))_k$ converges. Then $(x_{n_k})_k$ is bounded as well, so by compactness of T', there exists a subsequence $(x_{n_{k_l}})_l$ of $(x_{n_k})_k$ such that $(T'(x_{n_{k_l}}))_l$ converges. But then $(T(x_{n_{k_l}}))_l$ is a subsequence of a convergent sequence, and so it converges as well. Hence

$$\lim_{l\to\infty} (\lambda T(x_{n_{k_l}}) + T'(x_{n_{k_l}}))$$

$$= \lambda \lim_{l\to\infty} T(x_{n_{k_l}}) + \lim_{l\to\infty} T'(x_{n_{k_l}})$$

converges as well. We conclude that $\lambda T + T'$ is compact.

b) Let $T \in \mathbb{K}(X,Y)$. Let $(x_n)_n$ be bounded in X. Then there exists a subsequence $(x_{n_k})_k$ such that $(T(x_{n_k}))_k \to y \in Y$, say. By continuity of S, we get that $(S(T(x_{n_k})))_k \to S(y)$. Hence $(ST(x_{n_k}))_k$ is convergent, and so ST is compact.

c) Let $S \in \mathbb{K}(Y,Z)$. Let $(x_n)_n$ be bounded in X. Then since T is bounded, the sequence $(T(x_n))_n$ is bounded in Y. By compactness of S $(S(T(x_n)))_n$ has a convergent subsequence, and so ST must be compact.

d) Immediate from c).

Exercise 5.2 Let $\lambda \in \ell^\infty(\mathbb{N})$. Suppose for a contrapositive proof that $\lambda \notin c_0(\mathbb{N})$. Then there exists $\varepsilon > 0$ and a subsequence $(\lambda_{n_k})_k$ of $(\lambda_n)_n$ such that $|\lambda_{n_k}| \geq \varepsilon$ for all $k \in \mathbb{N}$. Define a sequence $(x^k)_{\ell=1}^\infty$ of elements in $\ell^p(\mathbb{N})$ by

$$(x^k)_r = \begin{cases} 1 & ; \ r = n_k \\ 0 & ; \ \text{otherwise}. \end{cases}$$

Then

$$(M_\lambda(x^k))_r = \begin{cases} \lambda_{n_k} & ; \ r = n_k \\ 0 & ; \ \text{otherwise}. \end{cases}$$

Note that $\|x^k\|_p = 1$ so that $(x^k)_{k=1}^\infty$ is bounded.
However, if $k \neq l$, then

$$\|M_\lambda(x^k) - M_\lambda(x^l)\|_p^p = \sum_{r \in \mathbb{N}} |\lambda_r (x^k)_r - \lambda_r (x^l)_r|^p$$

$$= |\lambda_{n_k} - 0|^p + |0 - \lambda_{n_l}|^p = |\lambda_{n_k}|^p + |\lambda_{n_l}|^p \geq 2\varepsilon^p,$$

This shows that $(M_\lambda(x^k))_{k=1}^\infty$ cannot have any convergent subsequence, which shows that M_λ is not compact.

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Problem 4 a) Let $x, y \in W$, say

$$x = \sum_{n=1}^{k} \alpha_n e_n \qquad y = \sum_{n=1}^{l} \beta_n e_n .$$

Then assume without loss of generality that $k \geq l$, and set $\beta_n = 0$ for $l < n \leq k$. Then for $\mu, \lambda \in \mathbb{C}$:

$$T_p(\mu x + \lambda y) = T_p\left(\sum_{n=1}^{k} (\mu\alpha_n + \lambda\beta_n) e_n\right) = \sum_{n=1}^{k} (\mu\alpha_n + \lambda\beta_n) e_{pn}$$

$$= \mu \sum_{n=1}^{k} \alpha_n e_{pn} + \lambda \sum_{n=1}^{l} \beta_n e_{pn} = \mu T_p(x) + \lambda T_p(y).$$

We also have

$$\| T_p(x) \|^2 = \left\| \sum_{n=1}^{k} \alpha_n e_{pn} \right\|^2 = \sum_{n=1}^{k} |\alpha_n|^2 = \|x\|^2$$

from which we conclude that T_p is bounded. Hence, by Theorem 3.3.2, there is a unique extension of T_p to H satisfying $S_p e_n = T_p e_n = e_{pn}$. $\forall n \in \mathbb{N}$.

b) (i) Let $m, n \in \mathbb{N}$. Then

$$\langle S_p^* S_q(e_m), e_n \rangle = \langle S_q(e_m), S_p(e_n) \rangle = \langle e_{qm}, e_{pn} \rangle$$

$$= \delta_{q^m, p^n}.$$

Since p and q are distinct prime numbers, it is impossible that $q^m = p^n$. Hence $\langle S_p^* S_q e_m, e_n \rangle = 0$ for all $m, n \in \mathbb{N}$, and so

$$\langle T(x), x \rangle = \left\langle T\left(\sum_n \langle x, e_n \rangle e_n\right), \sum_l \langle x, e_l \rangle e_l \right\rangle$$

$$= \sum_{n,l} \langle x, e_n \rangle \overline{\langle x, e_l \rangle} \langle T(e_n), e_l \rangle = 0 \quad \text{for all } x \in H.$$

By exercise 4.24c), we conclude that

$$S_p^* S_q = 0 .$$

Let $x \in H$. Then

$$\langle S_p x, S_p y \rangle = \left\langle S_p\left(\sum_n \langle x, e_n \rangle e_n \right), S_p\left(\sum_l \langle y, e_l \rangle e_l \right) \right\rangle$$

$$= \left\langle \sum_n \langle x, e_n \rangle e_{pn}, \sum_l \langle y, e_l \rangle e_{pl} \right\rangle$$

$$= \sum_{n, l \in \mathbb{N}} \langle x, e_n \rangle \overline{\langle y, e_l \rangle} \langle e_{pn}, e_{pl} \rangle$$

$$= \sum_{n \in \mathbb{N}} \langle x, e_n \rangle \overline{\langle y, e_n \rangle} = \langle x, y \rangle .$$

This shows that S_p is inner product preserving.

By Proposition 4.5.1, it follows that $S_p^* S_p = I$.

Problem 5

We begin by proving that for nonnegative, $\mathcal{A}$-measurable functions f on Ω, the following holds:

$$\int_\Omega f\,d\nu \leq \int_\Omega f\,d\mu. \qquad (*)$$

Begin by letting $f = \mathbb{1}_A$ for some $A \in \mathcal{A}$. Then

$$\int_\Omega f\,d\nu = \nu(A) \leq \mu(A) = \int_\Omega f\,d\mu.$$

Next, let f be simple, say $f = \sum_{j=1}^{k} c_j \mathbb{1}_{A_j}$ with $c_j \geq 0$, $A_j \in \mathcal{A}$.
Then

$$\int_\Omega f\,d\nu = \sum_{j=1}^{k} c_j \int_\Omega \mathbb{1}_{A_j}\,d\nu \leq \sum_{j=1}^{k} c_j \int \mathbb{1}_{A_j}\,d\mu = \int_\Omega f\,d\mu.$$

For a general $\mathcal{A}$-measurable function f, let $(s_n)_{n=1}^{\infty}$ be an increasing sequence of simple functions increasing pointwise to f. By the MCT, we get

$$\int_\Omega f\,dv = \lim_{n\to\infty} \int_\Omega s_n\,d\nu \leq \lim_{n\to\infty} \int_\Omega s_n\,d\mu = \int_\Omega f\,d\mu.$$

By exercise 2.13 b) from ELA, we have that $\mathcal{L}^2(\Omega,\mathcal{A},\mu) \subseteq \mathcal{L}^1(\Omega,\mathcal{A},\mu)$ with

$$(\Delta) \qquad \|f\|_1 \leq \mu(\Omega)^{\frac{1}{2}} \|f\|_2 \quad \text{for } f \in \mathcal{L}^2(\Omega,\mathcal{A},\mu).$$

If $f \in \mathcal{L}^2(\Omega,\mathcal{A},\mu)$, we get by $(*)$ and (Δ) that

$$\int_\Omega |f|\,d\nu \leq \int_\Omega |f|\,d\mu = \|f\|_1 \leq \mu(\Omega)^{\frac{1}{2}} \|f\|_2 < \infty.$$

This shows that for $f \in \mathcal{L}^2(\Omega, \mathcal{A}, \mu)$, f is absolutely integrable w.r.t. ν, and so the integral $\int_\Omega f \, d\nu$ is well-defined. Define a map

$$\phi : L^2(\Omega, \mathcal{A}, \mu) \longrightarrow \mathbb{C}$$
$$[f] \longmapsto \int_\Omega f \, d\nu .$$

We have to check that if $f, g \in \mathcal{L}^2(\Omega, \mathcal{A}, \mu)$ with $f = g$ μ-a.e., then $\int_\Omega f \, d\nu = \int_\Omega g \, d\nu$. If $f = g$ μ-a.e., then $\mu(N) = 0$, where

$$N = \{ x \in X : f(x) \neq g(x) \}.$$

But then $\nu(N) \leq \mu(N) = 0$, showing that $f = g$ ν-a.e. But then $\int_\Omega f \, d\nu = \int_\Omega g \, d\nu$, which is wanted. Now ϕ is linear by the linearity of the integral. Finally, if $f \in L^2(\Omega, \mathcal{A}, \mu)$, then

$$\left| \int_\Omega f \, d\nu \right| \overset{(*)}{\leq} \int_\Omega |f| \, d\nu \leq \int_\Omega |f| \, d\mu = \|f\|_1 \overset{(\Delta)}{\leq} \mu(\Omega)^{\frac{1}{2}} \|f\|_2$$

which shows that ϕ is bounded. By the Riesz representation theorem (Theorem 4.3.1.), there exists $[g] \in L^2(\Omega, \mathcal{A}, \mu)$ such that

$$\int_\Omega f \, d\nu = \phi([f]) = \langle [f], [g] \rangle$$

for all $f \in L^2(\Omega, \mathcal{A}, \mu)$. Letting $A \in \mathcal{A}$, we get, since $\mathbb{1}_A \in L^2(\mu)$:

$$\nu(A) = \int_\Omega \mathbb{1}_A \, d\nu = \phi([\mathbb{1}_A]) = \langle [\mathbb{1}_A], [g] \rangle = \int_\Omega \mathbb{1}_A \bar{g} \, d\mu = \int_\Omega \bar{g} \, d\mu.$$

Hence $\bar{g}$ is the function we have been looking for.