MAT3400/4400 Spring 2019

Exercises May 13 SOLUTIONS

Exercise 5.4 First, let T have rank-one. Then by exercise 4.18, $T = T_{x,y}$ for some $x, y \in H$.

Then $\langle T_{x,y}(u_n), u_n \rangle = \langle \langle u_n, x \rangle y, u_n \rangle = \langle u_n, x \rangle \langle y, u_n \rangle$.

Note that since $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2$ for every $x \in H$ (Parseval's theorem), the sequence $\{|\langle x, u_n \rangle|\}_{n=1}^{\infty}$ must converge to 0. Consequently, we have that

$$\lim_{n\to\infty} \langle T_{x,y}(u_n), u_n \rangle = \left(\lim_{n\to\infty} \langle u_n, x \rangle\right)\left(\lim_{n\to\infty} \langle y, u_n \rangle\right) = 0.$$

Next, let F have finite rank. By exercise 4.18,

$T = \sum_{k=1}^{K} T_{x_k, y_k}$ for some $x_k, y_k \in H$. Then

$$\langle T(u_n), u_n \rangle = \left\langle \sum_{k=1}^{K} \langle u_n, x_k \rangle y_k, u_n \right\rangle = \sum_{k=1}^{K} \langle u_n, x_k \rangle \langle y_k, u_n \rangle$$

which goes to zero by the previous argument.

Finally, let $T \in K(H)$. Then $T = \lim_{n\to\infty} T_n$ where each T_n is finite-rank by Theorem 5.2.1. For all $m, n \in \mathbb{N}$, we have that

$$\begin{aligned}|\langle T(u_n), u_n \rangle| &= |\langle (T - T_m + T_m)(u_n), u_n \rangle| \\ &\leq |\langle (T - T_m)(u_n), u_n \rangle| + |\langle T_m(u_n), u_n \rangle| \\ &\leq \|T - T_m\| + |\langle T_m(u_n), u_n \rangle|.\end{aligned}$$

Let $\varepsilon > 0$. Then there exists $M \in \mathbb{N}$ with

$$m \geq M \Rightarrow \|T - T_m\| < \frac{\varepsilon}{2}.$$

There also exists $N \in \mathbb{N}$ with

$$n \geq N \implies |\langle T_M(u_n), u_n \rangle| < \frac{\varepsilon}{2}$$

since T_M is finite-rank. Thus, if $n \geq N$, then

$$|\langle T(u_n), u_n \rangle| \leq \|T - T_M\| + |\langle T_M(u_n), u_n \rangle|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that $(T(u_n), u_n) \to 0$.

Exercise 5.5

Let $P \in \mathcal{B}(H)$ satisfy $P^2 = P$. If P has finite rank then P is compact (Corollary 5.1.6.).
Conversely, suppose P is compact. Let $M = \mathrm{Im}(P) = P(H)$.
Then if $y \in P(H)$, then $y = P(x)$ for some $x \in H$, and so $P(y) = P^2(x) = P(x) = y$, i.e. $P(y) = y$. Hence, the restriction of P to M is the identity map. But then $id_M : M \to M$ is a compact operator. In particular, $P(M_1) = M_1$ is a relatively compact subset of M by Proposition 5.1.2. (Here, M_1 is the unit ball of M).
It follows that M_1 is relatively compact and closed, hence compact. But by Exercise 3.2., the unit ball is always noncompact in infinite-dimensional spaces. It follows that $M = P(H)$ is finite-dimensional. Hence P is finite-rank.

Exercise 5.6

b) Let $T \in \mathcal{HS}(H)$, $S \in \mathcal{B}(H)$. Let $\{u_n\}_{n \in \mathbb{N}}$ be an orthonormal basis for H. Then

$\sum_{n \in \mathbb{N}} \|T(u_n)\|^2 < \infty$ since T is Hilbert-Schmidt,

and so

$$(*) \quad \sum_{n \in \mathbb{N}} \|ST(u_n)\|^2 \leq \sum_{n \in \mathbb{N}} \|S\|^2 \|T(u_n)\|^2 \leq \|S\|^2 \sum_{n=1}^{\infty} \|T(u_n)\|^2 < \infty .$$

Thus, ST is Hilbert-Schmidt. Moreover, the proof of lemma 5.2.5. shows that

$$T \in \mathcal{HS}(H) \implies T^* \in \mathcal{HS}(H).$$

So $T^* \in \mathcal{HS}(H)$, and by what we already proved, $S^*T^* \in \mathcal{HS}(H)$. But then

$$TS = (S^*T^*)^* \in \mathcal{HS}(H) \text{ as well.}$$

In fact, by lemma 5.2.5., we have $\|T^*\|_2 = \|T\|$, and so $(*)$ gives both

$$\|ST\|_2^2 \leq \|S\|^2 \|T\|_2^2 \quad \text{and}$$

$$\|TS\|_2 = \|S^*T^*\|_2 \leq \|S^*\| \, \|T^*\|_2 = \|S\| \, \|T\|_2 .$$

c) Using the polarization identity, we have that

$$(*) \quad \sum_{j\in J} \langle T(u_j), T'(u_j)\rangle = \sum_{j\in J}\sum_{k=0}^{3} \frac{i^k}{4} \| T(u_j) + i^k T'(u_j)\|^2$$

$$= \sum_{k=0}^{3}\sum_{j\in J} \frac{i^k}{4} \|(T+i^kT')(u_j)\|^2 = \sum_{k=0}^{3} \frac{i^k}{4} \| T + i^k T'\|_2^2$$

Since $HS(H)$ is a subspace of $B(H)$, we have that $T + i^k T' \in HS(H)$ for $k = 0, 1, 2, 3$, and so the sum $(*)$ converges.

Linearity in the first argument of $\langle\cdot,\cdot\rangle_2$ goes as follows: Let $T_1, T_2, T_3 \in HS(H)$, $\lambda, \mu \in \mathbb{C}$. Then

$$\langle \lambda T_1 + \mu T_2, T_3\rangle_2 = \sum_{j\in J} \langle (\lambda T_1 + \mu T_2)(u_j), T_3(u_j)\rangle$$

$$= \lambda \sum_{j\in J} \langle T_1(u_j), T_3(u_j)\rangle + \mu \sum_{j\in J} \langle T_2(u_j), T_3(u_j)\rangle$$

$$= \lambda \langle T_1, T_3\rangle_2 + \mu \langle T_2, T_3\rangle_2.$$

Conjugate-symmetry: $\overline{\langle T_1, T_2\rangle_2} = \overline{\sum_{j\in J} \langle T_1(u_j), T_2(u_j)\rangle}$

$$= \sum_{j\in J} \overline{\langle T_1(u_j), T_2(u_j)\rangle} = \sum_{j\in J} \langle T_2(u_j), T_1(u_j)\rangle = \langle T_2, T_1\rangle_2.$$

Positive-definiteness:

$\sum_{j\in J} \langle T(u_j), T(u_j)\rangle \geq 0$ since $\langle T(u_j), T(u_j)\rangle \geq 0$ for each j. Moreover, if $\sum_{j\in J} \langle T(u_j), T(u_j)\rangle = 0$ then $\|T(u_j)\|^2 = \langle T(u_j), T(u_j)\rangle = 0$ for all j, giving $T(u_j)=0$ for all $j\in J$. Since $\{u_j\}_{j\in J}$ is an orthonormal basis, we conclude that $T=0$.

This shows that $\langle\cdot,\cdot\rangle_2$ is an inner product on $\mathcal{HS}(H)$. Finally, note that

$$\langle T,T\rangle_2 = \sum_{j\in J}\langle T(u_j),T(u_j)\rangle = \sum_{j\in J}\|T(u_j)\|^2 = \|T\|_2^2$$

which shows that the norm coming from $\langle\cdot,\cdot\rangle_2$ is the Hilbert-Schmidt norm.

d) Let $\{T_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{HS}(H)$. Then for all $\varepsilon>0$ $\exists N$ $\forall m,n\geq N$: $\|T_m-T_n\|_2<\varepsilon$.

But since $\|T_m-T_n\|\leq\|T_m-T_n\|_2$ for all m,n, we get that $\{T_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{B}(H)$. Since $\mathcal{B}(H)$ is complete, there exists $T\in\mathcal{B}(H)$ such that $\|T-T_n\|\to 0$.

Now let $\varepsilon>0$. Let N be such that $\forall m,n\geq N$ we have $\|T_m-T_n\|_2^2<\varepsilon$. For any $J\in\mathbb{N}$, we have for $n\geq N$

$$\sum_{j=1}^{J}\|T(u_j)-T_n(u_j)\|^2 = \lim_{m\to\infty}\sum_{j=1}^{J}\|T_m(u_j)-T_n(u_j)\|^2$$

$$\leq \lim_{m\to\infty}\sum_{j=1}^{\infty}\|T_m(u_j)-T_n(u_j)\|^2<\varepsilon.$$

Since $J\in\mathbb{N}$ was arbitrary, this shows that for $n\geq N$

$$\sum_{j=1}^{\infty}\|T(u_j)-\|T_n(u_j)\|^2 = \lim_{J\to\infty}\sum_{j=1}^{J}\|T(u_j)-T_n(u_j)\|^2<\varepsilon.$$

In other words, we have that

$$\sum_{j=1}^{\infty} \|T(u_j) - T_n(u_j)\|^2 \to 0 \quad \text{as } n \to \infty. \qquad (*)$$

Now for any $n \in \mathbb{N}$ we have

$$\left(\sum_{j=1}^{\infty} \|T(u_j)\|^2 \right)^{\frac{1}{2}} = \left(\sum_{j=1}^{\infty} \|(T - T_n + T_n)(u_j)\|^2 \right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{j=1}^{\infty} \|T(u_j) - T_n(u_j)\|^2 \right)^{\frac{1}{2}} + \left(\sum_{j=1}^{\infty} \|T_n(u_j)\|^2 \right)^{\frac{1}{2}}.$$

By $(*)$, the first term above goes to zero as $n \to \infty$. The second term above is $\|T_n\|_2$, and since $\{T_n\}_n$ is Cauchy in $HS(H)$, the sequence $\{\|T_n\|_2\}_n$ is bounded. Thus, we have that

$$\sum_{j=1}^{\infty} \|T(u_j)\|^2 < \infty$$

which shows that $T \in HS(H)$. Now, $(*)$ gives us that $\|T - T_n\|_2 \to 0$.

Exercise 5.8

Since $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence and $T \in \mathcal{K}(H)$, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_n$ such that $\{T(x_{n_k})\}_k \to y$, say. Then we have that

$\|T(x_{n_k}) - \lambda x_{n_k}\| \to 0$ as well, and so

$$\|y - \lambda x_{n_k}\| \leq \|y - T(x_{n_k})\| + \|T(x_{n_k}) - \lambda x_{n_k}\| \to 0$$

Hence $\{\lambda x_{n_k}\}_k \to y$. Let $x := \lambda^{-1} y$. Then

$$T(x) = T(\lambda^{-1} y) = \lambda^{-1} T(y) = \lambda^{-1} \lim_{k\to\infty} T(\lambda x_{n_k})$$
$$= \lim_{k\to\infty} T(x_{n_k}) = y = \lambda x.$$

This shows that λ is an eigenvalue of T, with eigenvector x.

Exercise 5.10

a) We can write G as the composition

$$\mathbb{R}^2 \xrightarrow{m} \mathbb{R} \xrightarrow{\tilde{g}} \mathbb{C}$$

where $m(s,t) = s - t$. Since these two maps are continuous, it follows that G is continuous.

b) By Example 4.4.5, an integral operator T_K is self-adjoint if and only if K is real-valued. In our case, this means that

$$\forall s,t \in [-\pi,\pi]: \quad \tilde{g}(s-t) \in \mathbb{R}$$

In particular, setting $t=0$, we need $\tilde{g}(s) \in \mathbb{R}$ $\forall s \in [-\pi,\pi]$. Thus we see that T_G is self-adjoint $\Leftrightarrow$ g is real-valued.

c) $$T_G(e_k)(s) = \int_{-\pi}^{\pi} \tilde{g}(s-t) e^{ikt} \, dt$$

[substituting $t' = s-t$] $\quad dt' = -dt$ $\quad t=-\pi \rightsquigarrow t' = s+\pi$ | $t=\pi \rightsquigarrow t' = s-\pi$

$$= -\int_{s+\pi}^{s-\pi} \tilde{g}(t') e^{ik(s-t')} \, dt'$$

Periodicity of $\tilde{g}$

$$= \int_{s-\pi}^{s+\pi} \tilde{g}(t') e^{iks} e^{-ikt'} \, dt' \overset{\downarrow}{=} e^{iks} \int_{-\pi}^{\pi} \tilde{g}(t') e^{-ikt'} \, dt'$$

$$= e^{iks} \langle g, e_k \rangle = \hat{g}(k) e_k(s).$$

Thus $T_G(e_k) = \hat{g}(k) e_k$, which shows that e_k is an eigenvector of T_G with eigenvalue $\hat{g}(k)$.

Since $\{e_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis for H consisting of eigenvectors of T_G, we deduce that T_G is diagonalizable.

d) ~~By Example 5.2.11., we have that~~

$$\cancel{\|T_G\|_2^2 = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |G(s,t)|^2 \, ds \, dt}$$

By Parseval's formula, we get

$$\|T_G\|^2 = \sum_{k\in\mathbb{Z}} \|T_G(e_k)\|^2 = \sum_{k\in\mathbb{Z}} \|\langle g, e_k\rangle e_k\|^2$$

$$= \sum_{k\in\mathbb{Z}} |\langle g, e_k\rangle|^2 = \|g\|_2^2.$$

Exam December 2 2016

Problem 2 a) Suppose $a \in (0, \frac{1}{p})$. Then

$$\int_{(0,1]} |g_a|^p \, d\mu = \int_{(0,1]} \frac{1}{x^{pa}} \, d\mu(x).$$

The sequence $\left\{ \frac{1}{x^{pa}} \cdot \mathbb{1}_{[\frac{1}{n},1]}(x) \right\}_n$ is a pointwise increasing sequence of measurable functions converging pointwise to $\frac{1}{x^{pa}}$. By the MCT:

$$\int_{(0,1]} |g_a|^p \, d\mu = \lim_{n\to\infty} \int_{[\frac{1}{n},1]} \frac{1}{x^{pa}} \, d\mu(x) = \lim_{n\to\infty} \int_{1/n}^{1} x^{-pa} \, dx$$

Rewrite as Riemann integral.

Since $0 < a < \frac{1}{p}$

$$= \lim_{n\to\infty} \frac{1}{-pa+1} x^{-pa+1} \Big|_{x=1/n}^{x=1}$$

$$= \lim_{n\to\infty} \frac{1}{-pa+1} \left(1 - \left(\frac{1}{n}\right)^{-pa+1}\right) = \frac{1}{-pa+1} = \frac{1}{1-pa}$$

This shows that $g_a \in \mathcal{L}^p$.

b) Since $|\ln(x)| \le \frac{1}{a x^a}$ for $0 < x \le 1$, we have that, choosing some $a \in (0, \frac{1}{p})$:

$$\int_{(0,1]} |h|^p \, d\mu \le \int_{(0,1]} \frac{1}{a^p x^{pa}} \, d\mu(x) = \frac{1}{a^p} \int_{(0,1]} |g_a|^p < \infty$$

By a)

c) Since $\lim_{x \to 0^+} \ln(x) = -\infty$, we have that $\lim_{x \to 0^+} |\ln(x)| = \infty$. Hence, since $\ln$ is continuous, there is no $K \geq 0$ for which $|\ln(x)| \leq K$ for all $x \in (0,1]$, and so $h \notin \mathcal{L}^\infty$.

Problem 3 a) Let $n \in \mathbb{N}$. Then

$$\int_{[0,\infty)} |f_n| \, d\lambda = \int_{[0,\infty)} |e^{-x} \cos(\tfrac{x}{n})| \, d\lambda(x) \leq \int_{[0,\infty)} e^{-x} \, d\lambda(x)$$

$$\underset{\substack{\uparrow \\ \text{Using} \\ \text{Riemann} \\ \text{integral}}}{=} \lim_{n \to \infty} \int_0^n e^{-x} \, dx = \lim_{n \to \infty} (-e^{-x}) \Big|_{x=0}^{x=n} = \lim_{n \to \infty} (-e^{-n} + 1)$$

$= 1$. Hence f_n is λ-integrable.

b) We have that

$$\lim_{n \to \infty} f_n(x) = e^{-x} \lim_{n \to \infty} \cos(\tfrac{x}{n}) = e^{-x} \cos(0) = e^{-x}$$

for all $x \in [0,\infty)$. Furthermore, as we saw in a), $|f_n(x)| \leq e^{-x}$ for all $x \in [0,1)$, and $g(x) = e^{-x}$ is λ-integrable. It follows from the DCT that

$$\lim_{n \to \infty} \int_{[0,\infty)} f_n \, d\lambda = \int_{[0,\infty)} e^{-x} \, d\lambda(x) \underset{\substack{\uparrow \\ \text{from a)}}}{=} 1$$