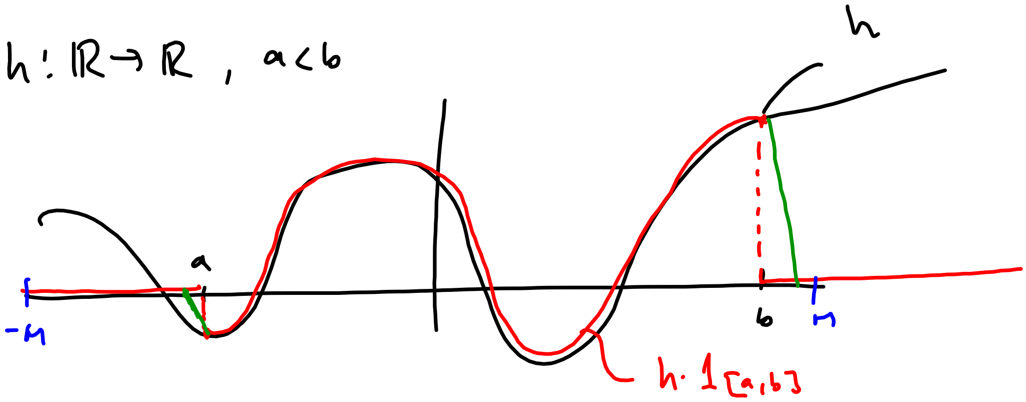
Littlewood's second principle (2.2 in Brevig's notes)

Def. $f:\mathbb{R}\to\mathbb{R}$ is called compactly supported if there exists some $M>0$ s.t. $f(x)=0$ whenever $|x|\geq M$

Ex $h:\mathbb{R}\to\mathbb{R}$, $a<b$



Then $f=h\cdot 1_{[a,b]}$ is comp. supported since $f(x)=h(x)\underbrace{1_{[a,b]}(x)}_{\substack{\Vert \\ 0 \\ \text{when } |x|\geq M}}=0$

Theorem (Littlewood's 2. principle): Assume $f:\mathbb{R}\to\mathbb{R}$ is Leb. integrable, and $\varepsilon>0$. Then there exists a continuous $g:\mathbb{R}\to\mathbb{R}$ which is comp. supported s.t.

$$\int_{\mathbb{R}}|f-g|\,d\mu<\varepsilon.$$

Idea: Approximate f by a simple function h. Then
— " — h by a step — " — k. Then
— " — k by a cont. comp. supp. g.

Note: A function $k:\mathbb{R}\to\mathbb{R}$ is called here a step function when $k=\sum_{j=1}^{J}c_j 1_{(a_j,b_j]}$ where $c_j\neq 0$, $-\infty<a_j<b_j<\infty$ for all $j=1,\dots,J$ $(J<\infty)$

Prop. 1 Let $(X, \mathcal{A}, \mu)$ be a measure space. Assume $f: X \to \mathbb{R}$ is integrable and $\varepsilon > 0$. Then there is a simple ~~function~~ integrable $h: X \to \mathbb{R}$ s.t. $\int_{\mathbb{R}} |f - h| \, d\mu < \varepsilon/3$

Proof. Write $f = f^+ - f^-$. Then we can find sequences $\{\varphi_n^+\}, \{\varphi_n^-\}$ of non-neg. simple functions on X such that

$$\varphi_n^+ \nearrow f^+, \quad \varphi_n^- \nearrow f^- \quad \text{pointwise on } X.$$

Set $h_n = \varphi_n^+ - \varphi_n^-$, $n \in \mathbb{N}$. Then $\{h_n\}_{n\in\mathbb{N}}$ is a seq. of simple functions on X s.t. $h_n \to f$ pointwise on X.

Moreover, we have

$$|h_n(x)| \leq \varphi_n^+(x) + \varphi_n^-(x) \leq f^+(x) + f^-(x) = |f(x)| \quad \forall x \in X.$$

So $|h_n| \leq |f|$ on X. As $\int_{\mathbb{R}} |f| \, d\mu < \infty$ (by assumption) we can apply the LDCT, and get that each h_n is integrable and

$$\int_{\mathbb{R}} |f - h_n| \, d\mu \longrightarrow 0 \text{ as } n \to \infty.$$

So we can pick N large enough so that $\int_{\mathbb{R}} |f - h_N| \, d\mu < \varepsilon/3$

We can now set $h := h_N$.

The second step is an application of Littlewood's first principle:

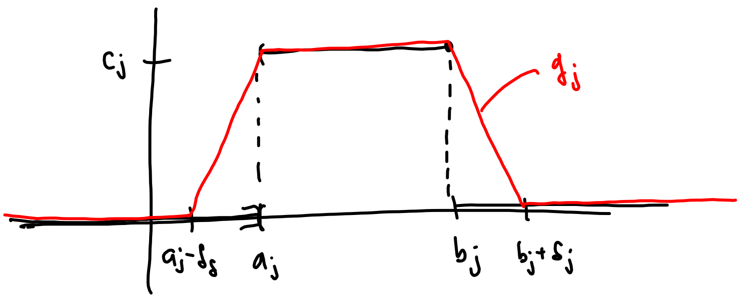
Prop. 2 Let $h: \mathbb{R} \to \mathbb{R}$ be a simple integrable function and $\varepsilon > 0$. Then we can find some step function $k: \mathbb{R} \to \mathbb{R}$ s.t.

$$\int_{\mathbb{R}} |h - k| \, d\mu < \varepsilon/3.$$

"Proof": Write h in standard form and approximate each of the measurable sets by finite union of intervals. We leave the details as an exercise.

Prop 3 Suppose $k: \mathbb{R} \to \mathbb{R}$ is a step function, and $\varepsilon > 0$. Then there exists $g: \mathbb{R} \to \mathbb{R}$ continuous and comp. supported s.t.
$$\int_{\mathbb{R}} |k - g| \, d\mu < \varepsilon/3$$

Proof Consider $k = \sum_{j=1}^{J} c_j \mathbb{1}_{(a_j, b_j]}$ (as in the def.)



Set $\delta_j := \dfrac{\varepsilon}{4J|c_j|}$ for each j.

Let g_j be the function whose graph is as above, for each j.

Set $g := g_1 + \cdots + g_J$, which is continuous and comp. supported (check!)

Then
$$\int_{\mathbb{R}} |k-g| \, d\mu = \int_{\mathbb{R}} \left| \sum_{j=1}^{J} c_j \mathbb{1}_{(a_j,b_j]} - \sum_{j=1}^{J} g_j \right| d\mu$$
$$\leq \int \sum_{j=1}^{J} |c_j \mathbb{1}_{(a_j,b_j]} - g_j| \, d\mu$$
the triangle ineq.
$$\leq \sum_{j=1}^{J} |c_j| \delta_j = \sum_{j=1}^{J} |c_j| \frac{\varepsilon}{4J|c_j|} = \varepsilon/4 < \varepsilon/3$$

Proof of the theorem: Let f be as in the theorem.

Then, using Prop. 1, we can find h simple, int., s.t. $\int_{\mathbb{R}} |f-h| \, d\mu < \varepsilon/3$

Next, — " — 2, — " — k step funct. s.t. $\int_{\mathbb{R}} |h-k| \, d\mu < \varepsilon/3$

Finally, — " — 3, — " — g cont. and comp. supp. s.t. $\int_{\mathbb{R}} |k-g| \, d\mu < \varepsilon/3$

Then we get
$$\int_{\mathbb{R}} |f-g| \, d\mu = \int_{\mathbb{R}} \underbrace{|f-h+h-k+k-g|}_{\leq |f-h|+|h-k|+|k-g|} \, d\mu$$
$$\leq \int_{\mathbb{R}} |f-h| \, d\mu + \int_{\mathbb{R}} |h-k| \, d\mu + \int_{\mathbb{R}} |k-g| \, d\mu$$
$$< \varepsilon.$$

The Fourier transform : a small appetizer

Let $f: \mathbb{R} \to \mathbb{R}$ be Lebesgue integrable.

Def. $\hat{f}: \mathbb{R} \to \mathbb{R}$ by $\boxed{\hat{f}(y) := \int_{\mathbb{R}} \underbrace{f(x)\, e^{-ixy}} \, d\mu(x) \quad \forall y \in \mathbb{R}}$

is a measurable function of x for each y, and it is integrable since $|f(x)\, e^{-ixy}| = |f(x)| \quad \forall x \in \mathbb{R}$

Note $\hat{f}$ is bounded since

$$|\hat{f}(y)| \leq \int_{\mathbb{R}} \underbrace{|f(x)\, e^{-ixy}|}_{= |f(x)|} \, d\mu(x)$$

$$= \int_{\mathbb{R}} |f| \, d\mu \quad \text{which does not depend on } y.$$

Riemann-Lebesgue Lemma $\boxed{\lim_{y \to \pm\infty} \hat{f}(y) = 0}$

"Sketch of proof": Assume $f = c\, 1_{[a,b]}$ $(a,b,c \in \mathbb{R},\ a<b)$

Then $\hat{f}(y) = \int_{\mathbb{R}} c\, 1_{[a,b]}(x)\, e^{-ixy}\, d\mu(x)$

$y \neq 0$

$$= \dots = c \int_a^b e^{-ixy}\, dx = c \left[\frac{1}{-iy} e^{-ixy} \right]_{x=a}^{x=b}$$

$$= c\, i\, \frac{e^{-iby} - e^{-iay}}{y} \to 0 \text{ as } y \to \pm\infty.$$

For the general case, "use" Prop. 1 and 2 (Exercise).

2.3 Littlewood's third principle:

Theorem ("Egorov's theorem"): Assume $(X, \mathcal{A}, \mu)$ is a finite measure space, i.e. $\mu(X) < \infty$. Let $\{f_n\}$ and f be measurable functions on X such that $f_n \to f$ pointwise on X.
Then $f_n \to f$ "almost uniformly" on X, meaning that
[For each $\varepsilon > 0$, we can find $E \in \mathcal{A}$ (depending on ε) s.t. $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on E^c.]

Proof. For $k, n \in \mathbb{N}$, define

$$E_{k,n} = \bigcup_{m=n}^{\infty} \{x \in X : |f_m(x) - f(x)| \geq \tfrac{1}{k}\} \in \mathcal{A}.$$

For k fixed, it is clear that $E_{k,1} \supseteq E_{k,2} \supseteq \cdots \supseteq E_{k,n} \supseteq \cdots \supseteq \bigcap_{n=1}^{\infty} E_{k,n}$

and, if $x \in E_{k,n}$ for all $n \in \mathbb{N}$, then

$$|f_n(x) - f(x)| \geq \tfrac{1}{k} \text{ for all } n \in \mathbb{N},$$

which is impossible (since $f_n(x) \to f(x)$).

Thus we have $\bigcap_{n=1}^{\infty} E_{k,n} = \emptyset$.

Now, $\mu(E_{k,1}) < \infty$, so using cont. from above, we get

$$\lim_{n\to\infty} \mu(E_{k,n}) = \mu\Big(\bigcap_{n=1}^{\infty} E_{k,n}\Big) = \mu(\emptyset) = 0$$

Now, Given $\varepsilon > 0$ and $k \in \mathbb{N}$, we can therfore find some $n_k \in \mathbb{N}$

s.t. $\mu(E_{k,n_k}) < \varepsilon/2^k$.

Let $E := \bigcup_{k=1}^{\infty} E_{k,n_k} \in \mathcal{A}$. Then $\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_{k,n_k}) < \varepsilon$

When $x \in E^c$, we have $x \notin E_{k,n_k}$ for all $k \in \mathbb{N}$

which gives that $|f_n(x) - f(x)| < \frac{1}{k}$ when $n > n_k$.

This we see that $f_n \to f$ uniformly on E^c.