Spaces (7.1)

Introduction : Why more integration theory?

Riemann integration has some defects:

Consider f. ex. $\mathcal{R}$ = all real-valued functions on $[0,1]$ which are Riemann-integrable.

Then $\mathcal{R}$ is too "small" in some aspects.

F. ex. let $f(x) = \begin{cases} 1, & x \in [0,1] \text{ } x \text{ rational} \\ 0, & \text{otherwise} \end{cases}$

Then $\underline{f \notin \mathcal{R}}$.

Moreover, if $q_1, q_2, \ldots, q_n, \ldots$ is a listing of all the rational numbers in $[0,1]$, and

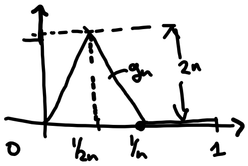
$$f_n(x) := \begin{cases} 1 & \text{if } x \in \{q_1, \ldots, q_n\}, \\ 0 & \text{otherwise} \end{cases}$$

for $x \in [0,1]$, then $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of functions in $\mathcal{R}$ such that

$f_n \to f$ pointwise on $[0,1]$.

Since $f \notin \mathcal{R}$, we see that <u>$\mathcal{R}$ is not "closed" under pointwise limits</u>.

The interchange of limits and integrals is also problematic:



$$\int_0^1 g_n(x)\,dx = 1 \quad \forall n$$

So $\lim\limits_{n\to\infty} \int_0^1 g_n(x)\,dx = 1$

But $\int_0^1 \underbrace{\lim_{n\to\infty} g_n(x)}_{\substack{\parallel \\ 0 \text{ for all } x}}\,dx = 0$

Hence $\lim\limits_{n\to\infty} \int_0^1 g_n(x)\,dx \neq \int_0^1 \lim g_n(x)\,dx$

Another problem is that f.ex. the space of continuous functions $C([0,1],\mathbb{R})$ equipped with the norm $\|f\|_1 = \int_0^1 |f(x)|\,dx$ is not complete (i.e. there are Cauchy sequences that do not converge).

Lebesgue-integration will fix all these problems!

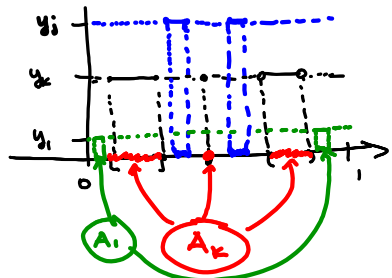
$\boxed{\text{Lebesgue's idea}}$: Instead of looking at partitions of the x-axis, look at partitions of the y-axis.

Recall: if $f: X \to Y$ is a function and $B \subseteq Y$, then $f^{-1}(B) = \{x \in X : f(x) \in B\}$

Let's consider $f: [0,1] \to \mathbb{R}$ taking only finitely many different values, $y_1, \dots, y_N$.

Set $\underbrace{f^{-1}(\{y_k\})}_{=:A_k} = \{x \in [0,1] : f(x) = y_k\}$,

so $[0,1] = \bigcup\limits_{k=1}^{N} A_k$. (disj. union)



The "area" under the graph of f will be $\sum\limits_{k=1}^{N} y_k \cdot \underbrace{(\text{the "size" of } A_k)}$

Each A_k can have a complicated structure, e.g. a countable union of intervals.
So we must give a meaning to what is the "size" of a set.

7.1 Measure spaces

Assume X is a set. Recall that $\mathcal{P}(X)$ denotes the set consisting of all subsets of X.
Thus $\mathcal{A} \subseteq \mathcal{P}(X)$ means that $\mathcal{A}$ is a collection of subsets of X.

Def We will say that $\mathcal{A} \subseteq \mathcal{P}(X)$ is a <u>σ-algebra</u> (on X) if the following conditions are satisfied:

(i) $\emptyset \in \mathcal{A}$
(ii) If $A \in \mathcal{A}$, then $A^c := X \setminus A \in \mathcal{A}$
(iii) If $\{A_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{A}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

The pair $(X, \mathcal{A})$ is called a <u>measurable space</u>.
The sets in $\mathcal{A}$ are often called $\mathcal{A}$-measurable (or just measurable).
The foll. properties are automatically satisfied:

a) $X \in \mathcal{A}$

→ since $X = \underbrace{\emptyset^c}_{\in \mathcal{A}}$ and $\emptyset \in \mathcal{A}$ using (i) and (ii)

b) $A_n \in \mathcal{A}$ for all $n \in \mathbb{N}$ $\Rightarrow \bigcap_{n\in\mathbb{N}} A_n \in \mathcal{A}$

→ Since $\bigcap_{n\in\mathbb{N}} A_n = \left(\left(\bigcap_{n\in\mathbb{N}} A_n\right)^c\right)^c$.

$$= \underbrace{\left(\underbrace{\bigcup_{n\in\mathbb{N}} \underbrace{A_n^c}_{\in\mathcal{A}}}_{\in\mathcal{A}}\right)^c}_{\in\mathcal{A}}$$

using (ii) and (iii)

c) If $A_1, \dots, A_n \in \mathcal{A}$, then $\bigcup_{j=1}^n A_j \in \mathcal{A}$ and $\bigcap_{j=1}^n A_j \in \mathcal{A}$.

Set $A_k = \emptyset$ for $k > n$
Then $\bigcup_{j=1}^n A_j = \bigcup_{j\in\mathbb{N}} A_j \in \mathcal{A}$
Since all A_j's belong to $\mathcal{A}$.

(For the second part, set $A_k := X$ for $k > n$.)

d) If $A, B \in \mathcal{A}$, then $A \setminus B \in \mathcal{A}$

Use that $A \setminus B = \underbrace{\underbrace{A}_{\in\mathcal{A}} \cap \underbrace{B^c}_{\in\mathcal{A}}}_{\in\mathcal{A}}$ (by using c))

Notation We set $\overline{\mathbb{R}}_+ := [0,\infty] := [0,\infty) \cup \{\infty\}$ and extend addition from $\mathbb{R}$ to $\overline{\mathbb{R}}_+$ by setting $\boxed{x+\infty := \infty + x := \infty \quad \forall x \in \overline{\mathbb{R}}_+}$.

Def. Assume $(X, \mathcal{A})$ is a measurable space. A function $\mu : \mathcal{A} \longrightarrow \overline{\mathbb{R}}_+$ is said to be a measure on $(X, \mathcal{A})$ when the foll. conditions hold:

(i) $\mu(\emptyset) = 0$

(ii) If $\{A_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{A}$ consisting of pairwise disjoint sets, then

$$\boxed{\mu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n)}$$

We say that $(X, \mathcal{A}, \mu)$ is a measure space