Measure spaces II.

Let X be a set, $X \neq \emptyset$.

Example $\mathcal{A} = \{\emptyset, X\}$ is obviously a σ-algebra (on X).

Set $\begin{cases} \mu(\emptyset) := 0 \\ \mu(X) := 1 \end{cases}$ (or any other value in $\overline{\mathbb{R}}_+$!)

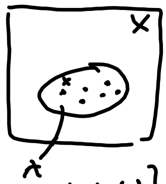
Then μ is a measure on $(X, \mathcal{A})$.

Example $\mathcal{A} = \mathcal{P}(X) =$ all the subsets of X.

(which is the largest possible σ-alg. on X).

Let $\rho : X \to \overline{\mathbb{R}}_+$ and let us think of ρ as a weight function on X.

We set $\boxed{\mu_\rho(A) = \sum_{x \in A} \rho(x)}$



where $\sum_{x \in A} \rho(A) := \begin{cases} \sup\{ \sum_{x \in F} \rho(x) : F \text{ finite subset of } A \} & \text{if } A \neq \emptyset \\ 0 & \text{if } A = \emptyset. \end{cases}$

Note we use here that if $S \subseteq \overline{\mathbb{R}}_+$, then

$$\sup S = \begin{cases} \infty & \text{if } \infty \in S \\ \infty & \text{if } \infty \notin S \text{ (so } S \subseteq \mathbb{R}_+) \text{ and } S \text{ is unbounded} \\ \sup S & \text{if } \infty \notin S \text{ and } S \text{ is bounded} \end{cases}$$

Then μ_ρ is a measure on $(X, \mathcal{P}(X))$

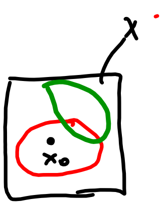
(Exercise)

- Choosing $\rho(x) = 1$ for all $x \in X$ gives that

$$\mu_\rho(A) = \sum_{x \in A} 1 = \begin{cases} |A| & \text{(the number of elements in } A) \text{ if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite.} \end{cases}$$

μ_ρ is called the counting measure on $(X, \mathcal{P}(X))$

- Picking $x_0 \in X$, we can choose



$$\rho(x) := \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{if } x \neq x_0. \end{cases}$$

Then we get $\mu_\rho(A) = \begin{cases} 1 & \text{if } x_0 \in A \\ 0 & \text{if } x_0 \notin A \end{cases}$

μ_ρ is called the point measure (or the Dirac measure) at x_0.

Note: If $\mathcal{B}$ is a σ-algebra on X, we can always restrict μ_ρ to $\mathcal{B}$ and get a measure on $(X, \mathcal{B})$.

<u>Example</u> Let $X = \mathbb{R}$. We will see in a couple of weeks that there exists a σ-algebra $\mathcal{L}$ on $\mathbb{R}$ which contains all open subsets and all closed subsets of $\mathbb{R}$ and a measure $\lambda : \mathcal{L} \to \overline{\mathbb{R}}_+$ (called the <u>Lebesgue measure</u>) such that

$$\boxed{\lambda([a,b]) = \lambda(a,b) = b-a} \quad \text{for all } a<b,\ a,b \in \mathbb{R}.$$

Elements of $\mathcal{L}$ are called <u>Lebesgue-measurable sets</u>

<u>Def</u> If $(X, \mathcal{A}, \mu)$ is a measure space s.t. $\mu(X) = 1$, then it is called a <u>probability space</u>.

In probability theory, X is the "sample space" of all possible outcomes, $\mathcal{A}$ consists of the measurable "events" and $\mu(A)$ is the probability that an outcome belongs to A.

<u>f.ex.</u> if we roll a dice N-times, we can take

$$X = \{1, 2, \dots, 6\}^N = \{(x_1, \dots, x_N) : 1 \le x_j \le 6, j = 1, \dots, N\}$$

$$\mathcal{A} = \mathcal{P}(X) \quad \text{and} \quad \mu(A) = \frac{|A|}{6^N}$$

A useful technical result:

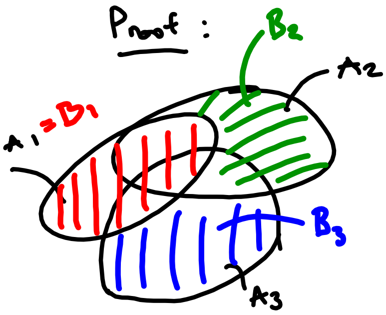
Lemma Assume $(X, \mathcal{A})$ is a measurable space, $\{A_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{A}$.
Then there exists a seq. $\{B_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}$ of disjoint sets in $\mathcal{A}$ such that

$B_n \subseteq A_n$ for all $n \in \mathbb{N}$ and

(*) $\bigcup_{n=1}^{m} B_n = \bigcup_{n=1}^{m} A_n$ for all $m \in \mathbb{N}$,

hence it satisfies that $\boxed{\bigcup_{n\in\mathbb{N}} B_n = \bigcup_{n\in\mathbb{N}} A_n}$

Proof:



Set
$$\begin{cases} B_1 := A_1 \in \mathcal{A} \\ B_2 = A_2 \setminus A_1 \in \mathcal{A} \\ B_3 = A_3 \setminus (A_1 \cup A_2) \in \mathcal{A} \\ \vdots \\ B_n = \underbrace{\underbrace{A_n}_{\in\mathcal{A}} \setminus \underbrace{(A_1 \cup \dots \cup A_{n-1})}_{\in\mathcal{A}}}_{\in\mathcal{A}} \quad \underline{n \in \mathbb{N}} \end{cases}$$

Clearly, we have $B_n \subseteq A_n$ for all n.

We prove (*) by induction on m:

- if $m=1$ then OK (since $A_1 = B_1$)
- Assume (*) is OK for some $m \in \mathbb{N}$.

Then
$$\bigcup_{n=1}^{m+1} B_n = \left(\bigcup_{n=1}^{m} B_n\right) \cup B_{m+1} = \left(\bigcup_{n=1}^{m} A_n\right) \cup \left(A_{m+1} \setminus \left(\bigcup_{k=1}^{m} A_n\right)\right)$$

using the ind. hypothesis.

$$= \bigcup_{n=1}^{m+1} A_n .$$

Moreover, the B_n's are pairwise disjoint:

Consider $1 \le j \le n-1$, $n \geq 2$. Then $B_j \cap B_n = \emptyset$ because
$$B_n = A_n \setminus \left(\bigcup_{k=1}^{n-1} A_k\right) \underset{(*)}{=} A_n \setminus \underbrace{\left(\bigcup_{k=1}^{n-1} B_k\right)}_{\text{contains } B_j} \subseteq A_n \setminus B_j = A_n \cap B_j^c \subseteq B_j^c$$

So $B_n \cap B_j = \emptyset$.

Properties of measures

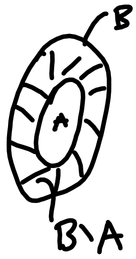
Proposition $(X, \mathcal{A}, \mu)$ measure space.

a) If $A_1, \dots, A_m$ are disjoint sets in $\mathcal{A}$, then
$$\mu\left(\bigcup_{k=1}^{m} A_k\right) = \sum_{k=1}^{m} \mu(A_k)$$

Consider the sequence $A_1, A_2, A_m, \emptyset, \emptyset, \dots$

b) If $A, B \in \mathcal{A}$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

c) —— " ——, and $\mu(A) < \infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$



d) If $\{A_n\}$ is a sequence in $\mathcal{A}$, then $\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$

Proof b) and c) We have $B = A \cup (B \setminus A)$, so (disjoint)

from a) we get that $\mu(B) = \mu(A) + \underbrace{\mu(B \setminus A)}_{\in [0,\infty)}$,

and we see that $\mu(A) \leq \mu(B)$.

If we assume now that $\mu(A) < \infty$, then we get that $\mu(B) - \mu(A) = \mu(B \setminus A)$, as desired.

d) Let the sequence $\{B_n\}$ be as in the previous lemma. Then we get
$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \mu\left(\bigcup_{n \in \mathbb{N}} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

Since $B_n \subseteq A_n$, so $\mu(B_n) \leq \mu(A_n)$ by b).

Example We roll a dice N-times again. What is the prob. of getting at least one 6?

Must find $\mu(A)$ where $A^c = \{(x_1, \dots, x_N) : 1 \leq x_j \leq 5, j = 1, \dots, N\}$

(and μ is as in the prev. example)

Since $|A^c| = 5^N$, we get $\mu(A^c) = \frac{5^N}{6^N} = \left(\frac{5}{6}\right)^N$

So $\mu(A) = \mu(X \setminus A^c) = \mu(X) - \mu(A^c) = \underline{\underline{1 - \left(5/6\right)^N}}$