7.1 (the end!)

Another important property of measure spaces:

Proposition Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}$.

a) Assume that $\{A_n\}$ is increasing, i.e. $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$.

Then $\boxed{\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{m\to\infty} \mu(A_m)}$

"Continuity of μ from below"

b) Assume that $\{A_n\}$ is decreasing, i.e. $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{N}$, and $\mu(A_1) < \infty$.

Then $\boxed{\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{m\to\infty} \mu(A_m)}$

"Continuity of μ from above"

Proof a) Set
$$\begin{cases} B_1 = A_1 \in \mathcal{A}. \\ B_2 = A_2 \setminus A_1 \in \mathcal{A} \\ \vdots \\ B_n = A_n \setminus A_{n-1} \in \mathcal{A} \\ \vdots \end{cases}$$
All the B_n's are disjoint.

Moreover ~~Then~~ $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$, so we get that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) = \lim_{m\to\infty} \underbrace{\sum_{n=1}^{m} \mu(B_n)}_{= \mu\left(\underbrace{\bigcup_{n=1}^{m} B_n}_{A_m}\right)}$$

b) Set $\tilde{A}_n = A_1 \setminus A_n$, $n \in \mathbb{N}$.

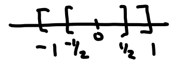
Then $\{\tilde{A}_n\}_{n\in\mathbb{N}}$ is increasing, and $\bigcup_{n=1}^{\infty} \tilde{A}_n = A_1 \setminus \left(\bigcap_{n=1}^{\infty} A_n\right)$, so we get that (from a))

$$\underbrace{\mu\left(\bigcup_{n=1}^{\infty} \tilde{A}_n\right)}_{\substack{\| \\ \underbrace{\mu\left(A_1 \setminus \left(\bigcap_{n=1}^{\infty} A_n\right)\right)}_{\substack{\| \\ \mu(A_1) - \mu\left(\bigcap_{n=1}^{\infty} A_n\right)}}}} = \underbrace{\lim_{m\to\infty} \underset{\substack{\| \\ \mu(A_1 \setminus A_m)}}{\mu(\tilde{A}_m)}}_{\substack{\| \\ \underbrace{\lim_{m\to\infty} (\mu(A_1) - \mu(A_m))}_{\substack{\| \\ \mu(A_1) - \lim_{m\to\infty} \mu(A_m)}}}}$$

Cancelling $\mu(A_1)$ on both sides, we get that
$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{m\to\infty} \mu(A_m).$$

Ex. Assume that μ is a measure on a σ-alg. on $\mathbb{R}$ containing all closed subsets of $\mathbb{R}$ and satisfying that $\mu\left(\left[-\frac{1}{n}, \frac{1}{n}\right]\right) = 1 + \frac{2}{n}$ for all $n \in \mathbb{N}$.

Then $\mu(\{0\}) = \mu\left(\bigcap_{n=1}^{\infty} \left[-\frac{1}{n}, \frac{1}{n}\right]\right)$



Using the Prop. above part b) with $A_n = [-1/n, 1/n]$; note $\mu(A_1) = 1 + \frac{2}{1} = 3 < \infty$.

$$= \lim_{m\to\infty} \mu\left(\left[-\frac{1}{m}, \frac{1}{m}\right]\right) = \underline{\underline{1}}.$$

7.2 Complete measures

Def. Let $(X, \mathcal{A}, \mu)$ be a measure space.

- A set $N \subseteq X$ is called a null set (w.r.t. μ) if there exists some $B \in \mathcal{A}$ such that $N \subseteq B$ and $\mu(B) = 0$.

Ex. If $N \in \mathcal{A}$ and $\mu(N) = 0$ then N is a null set (since we can take $B = N$).

Note that if N is a null set and $N \in \mathcal{A}$, then $\mu(N) = 0$ (because $0 \le \mu(N) \le \mu(B) = 0$ with B as above)

- We say that $(X, \mathcal{A}, \mu)$ is complete if all null sets (w.r.t. μ) belong to $\mathcal{A}$.

Ex. Consider $X \neq \emptyset$, $\mathcal{A} = \{\emptyset, X\}$, and μ = the zero-measure on $\mathcal{A}$, i.e. $\mu(\emptyset) = \mu(X) = 0$.
Then $(X, \mathcal{A}, \mu)$ is not complete (if X contains more than one element):

Pick any $E \subseteq X$, $\emptyset \neq E \neq X$.
~~Then~~ Since $\mu(X) = 0$, E is a null set (w.r.t. μ)
But $E \notin \mathcal{A}$.

It is sometimes useful to know that any measure space can be "completed":

Theorem $(X, \mathcal{A}, \mu)$ measure space.
Set $\mathcal{N} := \{N \subseteq X : N \text{ is a null set (w.r.t. } \mu)\}$,
$\bar{\mathcal{A}} := \{A \cup N : A \in \mathcal{A}, N \in \mathcal{N}\}$, and
define $\bar{\mu} : \bar{\mathcal{A}} \to \bar{\mathbb{R}}_+$ by $\bar{\mu}(A \cup N) := \mu(A)$ for all $A \in \mathcal{A}, N \in \mathcal{N}$.

Then $(X, \bar{\mathcal{A}}, \bar{\mu})$ is a complete measure space, $\mathcal{A} \subseteq \bar{\mathcal{A}}$ and $\bar{\mu}(A) = \mu(A)$ for all $A \in \mathcal{A}$.

(It can be shown that $\bar{\mu}$ is the unique extension of μ to $\bar{\mathcal{A}}$).

We won't need this result, so we skip the long, tedious proof. Read it yourself if you are in the right mood.

We note that $\bar{\mathcal{A}}$ contains $\mathcal{A}$ and $\mathcal{N}$ (because if $A \in \mathcal{A}$, $A = \underbrace{A}_{\in \mathcal{A}} \cup \underbrace{\emptyset}_{\in \mathcal{N}} \in \bar{\mathcal{A}}$, and if $N \in \mathcal{N}$, then $N = \underbrace{\emptyset}_{\in \mathcal{A}} \cup \underbrace{N}_{\in \mathcal{N}} \in \bar{\mathcal{A}}$).

In fact, $\bar{\mathcal{A}}$ is the smallest σ-algebra on X which contains $\mathcal{A}$ and $\mathcal{N}$:

Indeed, let $\mathcal{B}$ be any such σ-algebra. Then we have if $A \in \mathcal{A}$, $N \in \mathcal{N}$, then $A \in \mathcal{B}$ and $N \in \mathcal{B}$, so $A \cup N \in \mathcal{B}$. This shows that $\bar{\mathcal{A}} \subseteq \mathcal{B}$.

Given any $\mathcal{C} \subseteq \mathcal{P}(X)$, there is always a smallest σ-alg. on X which contains $\mathcal{C}$:

Proposition Assume $X \neq \emptyset$ and $\mathcal{C} \subseteq \mathcal{P}(X)$. Then there exists a smallest σ-alg. on X containing $\mathcal{C}$, which we denote by $\sigma(\mathcal{C})$ and call the σ-algebra generated by $\mathcal{C}$.

Proof Since $\mathcal{P}(X)$ is a σ-algebra on X which contains $\mathcal{C}$, we can define $\sigma(\mathcal{C})$ to be the intersection of all σ-alg. on X containing $\mathcal{C}$, i.e.

$$\sigma(\mathcal{C}) := \{ B \subseteq X \mid B \in \mathcal{A} \text{ for all } \sigma\text{-alg. } \mathcal{A} \text{ on } X \text{ containing } \mathcal{C} \}$$

First we check that $\sigma(\mathcal{C})$ is a σ-alg.:

(i) $\emptyset \in \mathcal{A}$ for all σ-alg. $\mathcal{A}$ containing $\mathcal{C}$, so $\emptyset \in \sigma(\mathcal{C})$.

(ii) Assume $B \in \sigma(\mathcal{C})$. Let $\mathcal{A}$ be any σ-alg. cont. $\mathcal{C}$. Then $B \in \mathcal{A}$. So $B^c \in \mathcal{A}$. This means that $B^c \in \sigma(\mathcal{C})$.

(iii) Assume $\{B_n\} \subseteq \sigma(\mathcal{C})$. Let $\mathcal{A}$ be any σ-alg. containing $\mathcal{C}$. Then $B_n \in \mathcal{A}$ for all $n \in \mathbb{N}$. So $\bigcup_{n=1}^{\infty} B_n \in \mathcal{A}$. This means that $\bigcup_{n=1}^{\infty} B_n \in \sigma(\mathcal{C})$.

Next, we check that $\sigma(\mathcal{C})$ contains $\mathcal{C}$:

If $C \in \mathcal{C}$, then $C \in \mathcal{A}$ for every σ-alg. $\mathcal{A}$ containing $\mathcal{C}$, so $C \in \sigma(\mathcal{C})$.

Finally, let $\mathcal{A}$ be any σ-alg. containing $\mathcal{C}$. Then, if $B \in \sigma(\mathcal{C})$, then $B \in \mathcal{A}$ (by def. of $\sigma(\mathcal{C})$), which shows that $\sigma(\mathcal{C}) \subseteq \mathcal{A}$.

Note that if $\mathcal{A}$ is a σ-alg. on X, then $\sigma(\mathcal{A}) = \mathcal{A}$.

Def. Let X be a metric space.
Let $\mathcal{G}_X$ denote the collection of all open subsets of X.
Then the σ-algebra $\mathcal{B}_X := \sigma(\mathcal{G}_X)$ generated by $\mathcal{G}_X$ is called the Borel σ-alg. on X

F.ex. $\mathcal{B}_{\mathbb{R}}$ is the Borel σ-alg. on $\mathbb{R}$ generated by all open subsets of $\mathbb{R}$ (equipped with its standard metric $d(x,y) = |x-y|$)

Note: If $\mathcal{F}_X$ denotes the collection of all closed subsets of a metric space X, then we have that

$$\mathcal{B}_X = \sigma(\mathcal{F}_X) \qquad (\text{Exercise!})$$

7.3 Measurable functions

We will consider functions taking values in

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$$ the extended real numbers

We extend ~~the~~ addition and multiplication to $\overline{\mathbb{R}}$ in the "obvious" way (cf. p.252). F. ex.

$$r \cdot \infty = \begin{cases} \infty & \text{if } r \in \overline{\mathbb{R}}_+ \\ -\infty & \text{if } (-r) \in \overline{\mathbb{R}}_+ \end{cases}.$$

But we consider "$\infty - \infty$" as undefined.

On the other hand, we set $\begin{cases} 0 \cdot \infty = \infty \cdot 0 := 0 \\ 0 \cdot (-\infty) = (-\infty) \cdot 0 := 0 \end{cases}$

(This is convenient in measure theory).