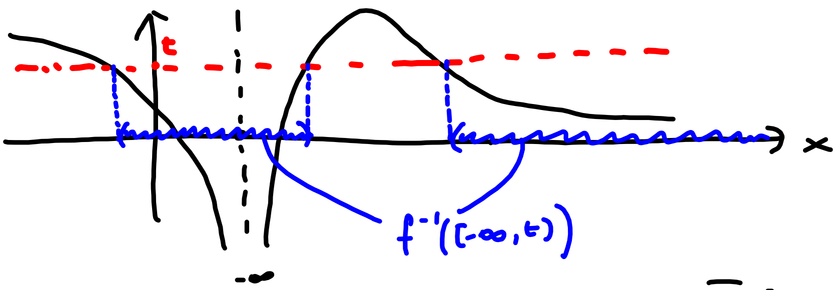
7.3 Measurable functions (ctd)

Let $(X,\mathcal{A})$ be a measurable space.
Recall that $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$.

Def. A function $f: X \to \overline{\mathbb{R}}$ is called <u>measurable</u> (w.r.t. $\mathcal{A}$), or $\mathcal{A}$-measurable, if

$$f^{-1}([-\infty,t)) \in \mathcal{A} \quad \text{for all } t \in \mathbb{R},$$

i.e. $\{x \in X \mid f(x) < t\} \in \mathcal{A}$ —″———.



Ex. 1 Assume $\mathcal{A} = \mathcal{P}(X)$. Then <u>any</u> $f: X \to \overline{\mathbb{R}}$ is measurable (w.r.t. $\mathcal{A}$).

Ex. 2 Assume $\mathcal{A} = \{\emptyset, X\}$. Then $f: X \to \overline{\mathbb{R}}$ is measurable (w.r.t. $\mathcal{A}$) $\iff$ f is constant.
(<u>Exercise!</u>).

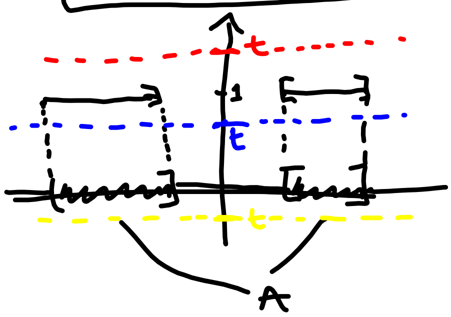
Ex. 3 Assume $A \subseteq X$ and define $1_A : X \to \mathbb{R} \subseteq \overline{\mathbb{R}}$ by

$$1_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in A^c. \end{cases}$$

<u>the indicator function of A</u>

also called <u>the characteristic function of A</u> and denoted by χ_A

$$\text{Then } \boxed{1_A \text{ is measurable (w.r.t. } \mathcal{A}) \iff A \in \mathcal{A}.}$$



Indeed,

$$(1_A)^{-1}([-\infty,t)) = \begin{cases} X & \text{if } t > 1. \\ A^c & \text{if } 0 < t \le 1 \\ \emptyset & \text{if } t \le 0 \end{cases}$$

So we see that all these sets belong to $\mathcal{A}$ $\iff$ $A^c \in \mathcal{A}$ $\iff$ $A \in \mathcal{A}$.

Prop. Assume $f : X \to \overline{\mathbb{R}}$ is measurable (w.r.t. $\mathcal{A}$)

① Let I be any interval, $I \subseteq \overline{\mathbb{R}}$, of the form (a,b), or $(a,b]$, or $[a,b)$, or $[a,b]$, or $\{a\}$ where $a,b \in \overline{\mathbb{R}}$, $a<b$.

Then $f^{-1}(I) \in \mathcal{A}$.

② We also have that $f^{-1}(B) \in \mathcal{A}$ for any $B \subseteq \mathbb{R}$ which is open or closed.

Note: More generally, we have that $f^{-1}(B) \in \mathcal{A}$ for any Borel subset B of $\mathbb{R}$. | Exercise!

Proof of ①:

i) $f^{-1}([-\infty,\infty]) = X \in \mathcal{A}$.

ii) $f^{-1}([-\infty,b]) = f^{-1}\left(\bigcap_{n=1}^{\infty}[-\infty, b+\frac{1}{n})\right)$
$b \in \mathbb{R}$

$$= \underbrace{\bigcap_{n=1}^{\infty} \underbrace{f^{-1}([-\infty, b+\tfrac{1}{n}))}_{\in \mathcal{A}}}_{\in \mathcal{A}}$$

since f is meas.

since $\mathcal{A}$ is a σ-alg.

iii) $f^{-1}([-\infty,\infty)) = f^{-1}\left(\bigcup_{n\in\mathbb{N}}[-\infty,n)\right)$

$$= \bigcup_{n=1}^{\infty} \underbrace{f^{-1}([-\infty,n))}_{\in \mathcal{A}} \in \mathcal{A}$$

iv) $f^{-1}([a,\infty]) = f^{-1}\big(\underbrace{\overline{\mathbb{R}} \setminus [-\infty,a)}_{\text{= the complement of } [-\infty,a) \text{ in } \overline{\mathbb{R}}}\big) = \underbrace{f^{-1}([-\infty,a))^c}_{\in \mathcal{A}}$.
$a \in \mathbb{R}$

v) $f^{-1}((a,\infty]) = f^{-1}(\overline{\mathbb{R}} \setminus [-\infty,a]) = \underbrace{f^{-1}([-\infty,a])^c}_{\in \mathcal{A} \text{ (using ii))}}$

vi) $f^{-1}((a,b)) = f^{-1}\big([-\infty,b) \cap (a,\infty]\big)$
$a,b \in \mathbb{R}$, $a<b$

$$= \underbrace{f^{-1}([-\infty,b))}_{\in \mathcal{A}} \cap \underbrace{f^{-1}((a,\infty])}_{\in \mathcal{A} \text{ (by v))}} \in \mathcal{A}.$$

..... all the remaining cases can be handled in a similar way. F. ex.

$f^{-1}(\{\infty\}) = f^{-1}(\overline{\mathbb{R}} \setminus [-\infty,\infty)) = f^{-1}([-\infty,\infty))^c \in \mathcal{A}$.
(using iii))

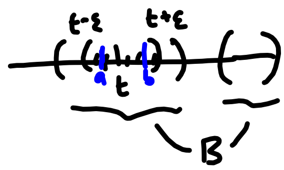
Proof of ②: Assume $B \subseteq \mathbb{R}$ is open, $B \neq \emptyset$.

Let $\mathcal{J} = \{(a,b) \text{ open interval} : a, b \in \mathbb{Q}\}$

Then $\mathcal{J}$ is countable $\to$ ($\underset{\text{open int. in } \mathcal{J}}{(a,b)} \longmapsto \underset{\text{in } \mathbb{Q}\times\mathbb{Q}}{(a,b) = \text{the point}}$ is injective; $\mathbb{Q}\times\mathbb{Q}$ is countable)

Let $t \in B$. Then there exists $(a,b) \in \mathcal{J}$ such that $t \in (a,b) \subseteq B$:

[Indeed, since B is open, we can find some $\varepsilon > 0$ s.t. $(t-\varepsilon, t+\varepsilon) \subseteq B$.
By density of $\mathbb{Q}$ in $\mathbb{R}$ we can find $a, b \in \mathbb{Q}$ s.t. $t-\varepsilon < a < t < b < t+\varepsilon$.
Then we have $t \in (a,b) \subseteq (t-\varepsilon, t+\varepsilon) \subseteq B$.



This means that $\mathcal{J}_B := \{(a,b) \in \mathcal{J} : (a,b) \subseteq B\} \subseteq \mathcal{J}$ is countable, non-empty and $B = \bigcup_{(a,b)\in\mathcal{J}_B} (a,b)$

This shows that $\underline{B = \dot{\bigcup_{n\in N}} (a_n, b_n)}$, where N is a countable subset of $\mathbb{N}$ and $a_n, b_n \in \mathbb{Q}$, $a_n < b_n$

Thus $$f^{-1}(B) = \underbrace{\bigcup_{n\in N} \underbrace{f^{-1}((a_n, b_n))}_{\in \mathcal{A} \text{ by } ①}}_{\in \mathcal{A}}.$$

Assume now B is closed, $B \subseteq \overline{\mathbb{R}}$.

Then
$$\begin{aligned} f^{-1}(B)^c &= f^{-1}(\overline{\mathbb{R}} \setminus B) \\ &= f^{-1}\big((\mathbb{R}\setminus B) \cup \{-\infty\} \cup \{\infty\}\big) \\ &= \underbrace{\underbrace{f^{-1}(\underbrace{\mathbb{R}\setminus B}_{\text{open in } \mathbb{R}})}_{\in \mathcal{A} \text{ (by the first part)}} \cup \underbrace{f^{-1}(\{-\infty\})}_{\in \mathcal{A}} \cup \underbrace{f^{-1}(\{\infty\})}_{\in \mathcal{A}}}_{\in \mathcal{A}} \quad (\text{by } ①) \end{aligned}$$

So $f^{-1}(B) = (f^{-1}(B)^c)^c \in \mathcal{A}$, as desired.

Prop. Let X be a metric space and let $\mathcal{A}$ be any σ-algebra on X containing the open sets of X (e.g. $\mathcal{A} = \mathcal{B}_X$).
Let $g: X \to \mathbb{R}$ be continuous.
Then g is measurable (w.r.t. $\mathcal{A}$).

Proof: Let $t \in \mathbb{R}$. Then $g^{-1}([-\infty, t)) = \underbrace{g^{-1}(\underbrace{(-\infty, t)}_{\text{open in } \mathbb{R}})}_{\substack{\text{open in } X \\ \text{since } g \text{ is continuous} \\ \text{(cf. 3.3.10 in [L])}}} \in \mathcal{A}$

So g is measurable.

Example Any real polynomial on $\mathbb{R}$ is measurable (w.r.t. any σ-alg. on $\mathbb{R}$ containing the open sets in $\mathbb{R}$, e.g. $\mathcal{B}_{\mathbb{R}}$)

Prop. Assume $f: X \to \mathbb{R}$ is measurable (w.r.t. $\mathcal{A}$) and $\phi: \mathbb{R} \to \mathbb{R}$ is continuous. Then $\phi \circ f$ is measurable.

Proof: Let $t \in \mathbb{R}$.

$$(\phi \circ f)^{-1}([-\infty, t)) = f^{-1}(\phi^{-1}([-\infty, t)))$$

$$= \underbrace{f^{-1}(\underbrace{\phi^{-1}((\underbrace{-\infty, t}_{\text{open}}))}_{\text{open } (\phi \text{ is continuous})})}_{\in \mathcal{A} \text{ (by the prev. prop.)}}$$

Note: This result also holds if ϕ is only assumed to be $\underbrace{\text{measurable w.r.t. } \mathcal{B}_{\mathbb{R}}}_{=: \text{ Borel measurable}}$.

Example If $f: X \to \mathbb{R}$ is measurable, then $f^n: X \to \mathbb{R}$, given by $f^n(x) := f(x)^n$ for $n \in \mathbb{N}$, is also measurable, since $f^n = \phi \circ f$ where $\phi: \mathbb{R} \to \mathbb{R}$ is def. by $\phi(t) = t^n$

Prop. Assume $f, g : X \to \mathbb{R}$ are measurable and $c \in \mathbb{R}$

Then $f+g$ and $c \cdot g$ are measurable $\to$ This means that the set of measurable functions from X to $\mathbb{R}$ is a real vector space.

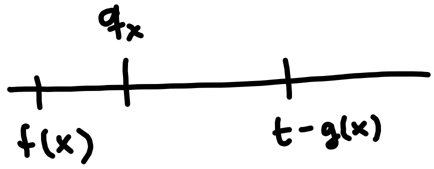
In particular, $f - g = f + (-1) \cdot g$ is measurable.

Moreover, fg is measurable.

Proof. Let $t \in \mathbb{R}$.

Then $(f+g)^{-1}([-\infty, t)) = (f+g)^{-1}((-\infty, t)) = \{x \in X : f(x) + g(x) < t\}$

$$= \{x \in X : f(x) < t - g(x)\}$$



For each x as above, we can find $q_x \in \mathbb{Q}$ s.t. $f(x) < q_x < t - g(x)$.

This gives that
$$(f+g)^{-1}([-\infty, t)) = \underbrace{\bigcup_{q \in \mathbb{Q}} \Big(\underbrace{\{x \mid f(x) < q\}}_{\in \mathcal{A}} \cap \underbrace{\{x \mid q < t - g(x)\}}_{= \underbrace{\{x \mid g(x) < t - q\}}_{\in \mathcal{A}}} \Big)}_{\in \mathcal{A}}.$$

So $f+g$ is measurable

$c \cdot g$ —"— (exercise!)

$$fg = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right) \to \text{is measurable!}$$

$(f+g)^2$, $(f-g)^2$ are measurable Using the prev. example