Measurable functions III

$(X, \mathcal{A})$ measurable space.

Consider a sequence $\{f_n\}$ of measurable functions (w.r.t. $\mathcal{A}$) where $f_n : X \to \overline{\mathbb{R}}$.

Prop. Let $k \in \mathbb{N}$. Let $\begin{cases} g_k := \inf_{n \geq k} f_n & : X \to \overline{\mathbb{R}} \\ h_k := \sup_{n \geq k} f_n & : X \to \overline{\mathbb{R}} \end{cases}$ be defined by

$$g_k(x) := \inf_{n \geq k} \{f_n(x)\}, \quad h_k(x) := \sup_{n \geq k} \{f_n(x)\}, \quad x \in X.$$

Then g_k and h_k are measurable.

Proof For each $t \in \mathbb{R}$, we have

$$(g_k)^{-1}([-\infty, t)) = \{x \in X : g_k(x) < t\}$$
$$= \{x \in X : \inf_{n \geq k} \{f_n(x)\} < t\}$$
$$\underbrace{= \bigcup_{n=k}^{\infty} \underbrace{\{x \in X : f_n(x) < t\}}_{= \underbrace{(f_n)^{-1}([-\infty, t))}_{\in \mathcal{A}}}}_{\in \mathcal{A}}.$$

$\supseteq$ is obvious.

$\subseteq$ If $\inf_{n \geq k} f_n(x) < t$ then there is $m \geq k$ such that $f_m(x) < t$ (because otherwise we would have $f_n(x) \geq t$ for all $n \geq k$, which would imply that $\inf_{n \geq k} f_n(x) \geq t$)

Hence g_k is measurable.

It then follows that $\underbrace{h_k = \sup_{n \geq k} f_n = -\underbrace{\inf_{n \geq k} (-f_n)}_{\text{is measurable by the argument above.}}}_{\text{is measurable.}}$

$-f_n$ are all measurable

Corollary 1. Let now $\liminf_{n\to\infty} f_n$ and $\limsup_{n\to\infty} f_n$ be defined

$$\liminf_{n\to\infty} f_n := \sup_{k\geq 1}\left(\inf_{n\geq k} f_n\right) \quad \left(= \sup_{k\geq 1} g_k \text{ with } g_k \text{ as in the prop.}\right)$$

$$\limsup_{n\to\infty} f_n := \inf_{k\geq 1}\left(\sup_{n\geq k} f_n\right) \quad \left(= \inf_{k\geq 1} h_k \text{ with } h_k \text{ —''—}\right)$$

(Note: $\liminf_{n\to\infty} f_n = \lim_{k\to\infty} g_k$ (because $\{g_k\}$ is increasing)
$\limsup_{n\to\infty} f_n = \lim_{k\to\infty} h_k$ (because $\{h_k\}$ is decreasing))

Then $\liminf_{n\to\infty} f_n$ and $\limsup_{n\to\infty} f_n$ are measurable.

Proof: Since each g_k and each h_k is measurable by the proposition, we get that $\liminf_{n\to\infty} f_n = \sup_{k\geq 1} g_k$ is measurable, $\limsup_{n\to\infty} f_n = \inf_{k\geq 1} h_k$ —''— again by the prop.

Corollary 2 Assume $f_n \to f$ pointwise on X (i.e. $f_n(x) \to f(x)$ as $n\to\infty$ for every $x\in X$)

Then f is measurable.

Proof The assumption gives that

$$f(x) = \lim_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x) \quad \left(= \liminf_{n\to\infty} f_n(x)\right)$$

for every $x\in X$, hence that $f = \limsup_{n\to\infty} f_n$, which is measurable by the previous corollary.

Corollary 3 Let $f: X\to\mathbb{R}$ be measurable. Then $|f|$ is measurable.

Proof Since $|t| = \max\{t,-t\}$ for each $t\in\mathbb{R}$, we ~~have~~ get that

$$|f| = \max\{f,-f\} = \sup_{n\geq 1}\{\underbrace{f_n}_{\text{are all measurable}}\} \quad \text{where } \begin{cases} f_1 = f \\ f_2 = -f \\ f_n = 0 & n\geq 3\end{cases}$$

is measurable by the prop.

(Alternatively, we could write $|f| = \underset{\text{continuous}}{\Phi}\circ \underset{\text{meas.}}{f}$ with $\Phi(t) = |t|$)

Assume $(X, \mathcal{A}, \mu)$ is a measure space.

Def. Let $f: X \to \overline{\mathbb{R}}$ be measurable. We will say that f is finite $\underbrace{\text{almost everywhere}}_{\text{a.e.}}$ (w.r.t. μ)

whenever $\mu(\{x \in X : f(x) = \pm\infty\}) = 0$.

belongs to $\mathcal{A}$ (cf. last week's notes)

If $g: X \to \overline{\mathbb{R}}$ is another measurable function, we will say that f and g are equal a.e. (w.r.t. μ)

whenever $\mu(\{x \in X \mid f(x) \neq g(x)\}) = 0$

this set belongs to $\mathcal{A}$ (Exercise!), in which case we write $f \sim g$ (w.r.t. μ)

Note: $\boxed{\sim \text{ is an equivalence relation on the set of } \overline{\mathbb{R}}\text{-valued measurable functions on } X. \text{ (Exercise)}}$

We will usually identify functions which are equal a.e.

Remark We will mostly consider functions which are finite a.e.

Let $f: X \to \overline{\mathbb{R}}$ be such a function, and

Set $E := \{x \in X \mid f(x) \in \mathbb{R}\} = X \setminus \{x \in X \mid f(x) = \pm\infty\}$

Define $\tilde{f}: X \to \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Then $\tilde{f}$ is measurable (cf. Exercise for Friday)

and $f \sim \tilde{f}$ (because $\{x \in X \mid f(x) \neq \tilde{f}(x)\} = E^c$
$= \{x \in X \mid f(x) = \pm\infty\}$ has measure 0 by assumption)

If g is another measurable function which is finite a.e., then it may happen that $f \pm g$ is not defined everywhere on X. But we can always consider $\tilde{f} \pm \tilde{g}$ instead.

Finally, note that if $f \sim g$, then we have that

$\tilde{f} \sim f \sim g \sim \tilde{g}$, i.e. $\tilde{f} \sim \tilde{g}$.

7.4 Integration of simple functions

$(X, \mathcal{A}, \mu)$ will be a fixed measure space.

Recall: If $A \subseteq X$, then $1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$, $x \in X$.
1_A is measurable $\Leftrightarrow$ $A \in \mathcal{A}$.

Def. A measurable function $f : X \to \mathbb{R}$ is called simple if f takes only finitely many values.

Example Assume $A \in \mathcal{A}$, $\emptyset \neq A \neq X$. Then 1_A is simple since it takes the values 0 and 1.

$1_\emptyset = 0$ is also simple.

$1_X = 1$ — " —

Moreover, a linear comb. of measurable indicator functions is also simple.

Note Let $f : X \to \mathbb{R}$ be a simple function and let $t_1, \dots, t_n$ be a listing of the distinct values of f.

Set $A_j = f^{-1}(\{t_j\}) = \{x \in X : f(x) = t_j\} \in \mathcal{A}$ for $j = 1, \dots, n$.

Then the A_j's are disjoint and we have

$$X = \bigcup_{j=1}^{n} A_j, \quad f = \sum_{j=1}^{n} t_j 1_{A_j}$$

This is called the standard form of f.

Note: It follows from this that the set of all simple functions is the span of $\{1_A : A \in \mathcal{A}\}$

Example $A \in \mathcal{A}$, $\emptyset \neq A \neq X$. Then $1_A = \underbrace{1 \cdot 1_A + 0 \cdot 1_{A^c}}_{\text{Stand. form of } 1_A}$

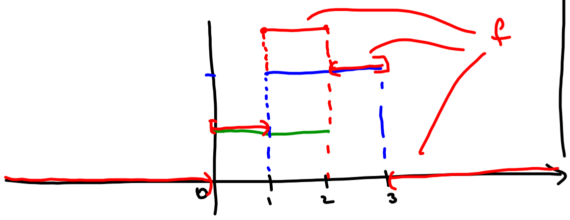
Note: When f takes the value 0 at some point, it is common to skip the zero-part in f's sf.

Note: Consider $f = \sum_{k=1}^{m} s_k 1_{B_k}$, where $B_1, \dots, B_m \in \mathcal{A}$ and $s_1, \dots, s_m \in \mathbb{R}$.

Then the standard form of f may not be $\sum_{k=1}^{n} s_k 1_{B_k}$

Example: Consider $X = \mathbb{R}$, $\mathcal{A} = \mathcal{P}(\mathbb{R})$, μ = any measure on $\mathbb{R}$.

Let $f = 1_{[0,2]} + 2 \cdot 1_{[1,3]}$



The stand. form of f is

$1 \cdot 1_{[0,1)} + 3 \cdot 1_{[1,2]} + 2 \cdot 1_{(2,3]} + 0 \cdot 1_{[0,3]^c}$