7.4 Integration of simple functions II

$(X, \mathcal{A}, \mu)$ measure space.

$\mathcal{S} :=$ all simple functions from X to $\mathbb{R}$

$= \operatorname{span}\{ 1_A : A \in \mathcal{A}\}$

Any $f \in \mathcal{S}$ has a standard form $\boxed{f = \sum_{k=1}^{m} t_k 1_{A_k}}$

where all t_k's are distinct real numbers, the A_k's are disjoint and belong to $\mathcal{A}$ and $X = \bigcup_{k=1}^{m} A_k$.

Recall that $\mathcal{M} = \{ f : X \to \mathbb{R} : f \text{ is measurable}\}$ is a vector space. Then $\mathcal{S}$ is clearly a subspace of $\mathcal{M}$.

($\mathcal{M}$ is also closed under multiplication. In fact,
$\mathcal{S}$ — " — (because $1_A 1_B = 1_{A\cap B}$))

We set $\mathcal{S}^+ := \{ f \in \mathcal{S} : f \geq 0 \}$ meaning that $f(x) \geq 0 \ \forall x \in X$.

Def. Let $f \in \mathcal{S}^+$. We define the integral of f w.r.t. μ by

$$\boxed{\int f \, d\mu := \sum_{k=1}^{m} t_k \, \mu(A_k)}$$

where $f = \sum_{k=1}^{m} t_k 1_{A_k}$ (stand. form)

Note: • We use the convention that $0 \cdot \infty = 0$.

• We sometimes write

$$\int_X f \, d\mu, \quad \text{or} \quad \int f(x) \, d\mu(x), \quad \text{or} \quad \int_X f(x) \, d\mu(x)$$

instead of $\int f \, d\mu$.

Example Let $A \in \mathcal{A}$, $\emptyset \neq A \neq X$. Then $\int 1_A \, d\mu = 1 \cdot \mu(A) + \underbrace{0 \cdot \mu(A^c)}_{0} = \mu(A)$

$1_A = 1 \cdot 1_A + 0 \cdot 1_{A^c}$ (st. form)

If $A = \emptyset$, then $\underset{0}{\int 1_\emptyset \, d\mu} = 0 \cdot \mu(X) = 0 = \mu(\emptyset)$.

$1_\emptyset = 0 \cdot 1_X$ (st. form)

Similarly, $\int 1_X \, d\mu = \mu(X)$.

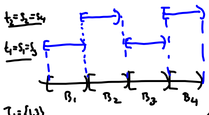
As we are going see, it is possible to compute the integral without having to find the standard form. A first step is:

Lemma Let $f \in \mathcal{S}^+$. Assume $f = \sum_{k=1}^{n} s_k 1_{B_k}$ where $s_1, \dots, s_n \in \mathbb{R}^+$, $B_1, \dots, B_n \in \mathcal{A}$, $B_j \cap B_k = \emptyset$ $(j \neq k)$, $\bigcup_{k=1}^{n} B_k = X$

(Note that the s_k's are not necessarily distinct).

Then $\int f \, d\mu = \sum_{k=1}^{n} s_k \, \mu(B_k)$.

Proof Let $f = \sum_{j=1}^{m} t_j 1_{A_j}$ (stand. form).



$I_1 = \{1,3\}$
$I_2 = \{2,4\}$
$A_1 = B_1 \cup B_3$
$A_2 = B_2 \cup B_4$

For $j = 1, \dots, m$, set $I_j = \{k \in \{1, \dots, n\} : s_k = t_j\}$

Then $A_j = \bigcup_{k \in I_j} B_k$ (disj. union) and $\bigcup_{j=1}^{m} I_j = \{1, 2, \dots, n\}$ (disj. union).

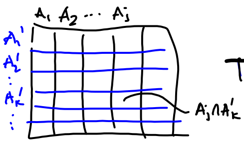
So
$$\int f \, d\mu = \sum_{j=1}^{m} t_j \, \mu(A_j) = \sum_{j=1}^{m} t_j \, \mu\Big(\bigcup_{k \in I_j} B_k\Big)$$
$$= \sum_{j=1}^{m} t_j \Big(\sum_{k \in I_j} \mu(B_k) \Big)$$
$$= \sum_{j=1}^{m} \sum_{k \in I_j} s_k \, \mu(B_k) = \sum_{k \in \bigcup_{j=1}^{m} I_j} s_k \, \mu(B_k)$$
$$= \sum_{k \in \{1, \dots, n\}} s_k \, \mu(B_k) = \sum_{k=1}^{n} s_k \, \mu(B_k).$$

Prop. Let $f, g \in \mathcal{S}^+$, $c \in \mathbb{R}^+$. Then

Exercise (i) $\int cf \, d\mu = c \int f \, d\mu$

(ii) $\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$.

Proof of (ii): Write
$$\begin{cases} f = \sum_{j=1}^{n} t_j 1_{A_j} & \text{(stand. form)} \\ g = \sum_{k=1}^{m} t'_k 1_{A'_k} & \text{(stand. form)} \end{cases}$$



Then we have
$$(*) \quad \boxed{\begin{cases} A_j = \bigcup_{k=1}^{m} (A_j \cap A'_k) \\ A'_k = \bigcup_{j=1}^{n} (A_j \cap A'_k) \end{cases} \text{ disj. unions}}$$

So we get
$$\int f \, d\mu + \int g \, d\mu = \sum_{j=1}^{n} t_j \, \mu(A_j) + \sum_{k=1}^{m} t'_k \, \mu(A'_k)$$
$$= \sum_{j=1}^{n} \sum_{k=1}^{m} t_j \, \mu(A_j \cap A'_k) + \sum_{k=1}^{m} \sum_{j=1}^{n} t'_k \, \mu(A_j \cap A'_k)$$
$$= \sum_{\substack{j=1,\dots,n \\ k=1,\dots,m}} (t_j + t'_k) \, \mu(A_j \cap A'_k)$$

On the other hand, using that $1_{\bigcup_{i=1}^{\ell} E_i} = \sum_{i=1}^{\ell} 1_{E_i}$ whenever the E_i's are disjoint,

we get that
$$f + g = \sum_{j=1}^{n} t_j 1_{A_j} + \sum_{k=1}^{m} t'_k 1_{A'_k}$$
using $(*)$
$$= \sum_{j=1}^{n} \sum_{k=1}^{m} t_j \, 1_{A_j \cap A'_k} + \sum_{k=1}^{m} \sum_{j=1}^{n} t'_k \, 1_{A_j \cap A'_k}$$
$$= \sum_{\substack{j=1,\dots,n \\ k=1,\dots,m}} (t_j + t'_k) \, 1_{A_j \cap A'_k}$$

Hence, using the previous lemma, we get that
$$\int (f+g) \, d\mu = \sum_{\substack{j=1,\dots,n \\ k=1,\dots,m}} (t_j + t'_k) \, \mu(A_j \cap A'_k)$$
$$= \int f \, d\mu + \int g \, d\mu$$

Remark: By induction we get that $\boxed{\int (f_1 + \dots + f_n) \, d\mu = \sum_{i=1}^{n} \int f_i \, d\mu}$ whenever $f_1, \dots, f_n \in \mathcal{S}^+$.

Corollary Let $f \in \mathcal{S}^+$, $f = \sum_{k=1}^m s_k \mathbb{1}_{B_k}$, where $s_1, \dots, s_m \in \mathbb{R}^+$ and $B_1, \dots, B_m \in \mathcal{A}$.

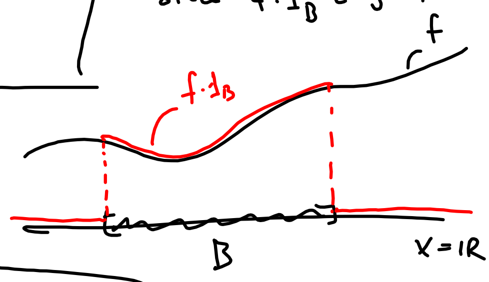
Then $$\boxed{\int f \, d\mu = \sum_{k=1}^m s_k \, \mu(B_k)}$$

Proof: $$\int f \, d\mu = \int \Big(\underbrace{\sum_{k=1}^m s_k \mathbb{1}_{B_k}}_{\in \mathcal{S}^+ \text{ for all } k} \Big) d\mu \underset{\uparrow \text{ the prev. remark}}{=} \sum_{k=1}^m \Big(\int s_k \mathbb{1}_{B_k} \, d\mu \Big)$$
$$= \sum_{k=1}^m s_k \, \mu(B_k).$$

Prop. $\boxed{\text{Let } f, g \in \mathcal{S}^+ \text{ and assume } g \leq f. \text{ Then } \int g \, d\mu \leq \int f \, d\mu}$

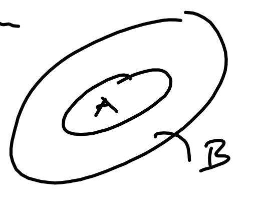
Proof We have $$\int f \, d\mu = \int \big(\underbrace{g}_{\in \mathcal{S}^+} + \underbrace{(f-g)}_{\in \mathcal{S}^+} \big) \, d\mu$$
$$= \int g \, d\mu + \underbrace{\int (f-g) \, d\mu}_{\geq 0} \geq \int g \, d\mu.$$

Notation If $f \in \mathcal{S}^+$ and $B \in \mathcal{A}$, then we set
$$\boxed{\int_B f \, d\mu := \int f \cdot \mathbb{1}_B \, d\mu}$$
which makes sense since $f \cdot \mathbb{1}_B \in \mathcal{S}^+$.



Exercise: $\boxed{\text{If } A, B \in \mathcal{A},\ A \cap B = \emptyset, \text{ then } \int_{A \cup B} f \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu} \ \boxed{\forall f \in \mathcal{S}^+}$

For example, if $A, B \in \mathcal{A}$ and $A \subseteq B$, then
$$\int_B f \, d\mu = \int_{A \cup (B \setminus A)} f \, d\mu = \int_A f \, d\mu + \int_{B \setminus A} f \, d\mu.$$



In the next section, we will need a couple of technical results.

Lemma: Assume $B \in \mathcal{A}$, $s \in \mathbb{R}^+$, and $\{f_n\}$ is an increasing sequence (i.e. $f_n(x) \leq f_{n+1}(x)$ $\forall n \in \mathbb{N}, \forall x \in X$) in $\mathcal{G}^+$ such that $\lim\limits_{n\to\infty} f_n(x) \geq s$ $\forall x \in B$.

Then $\boxed{\lim\limits_{n\to\infty} \int_B f_n \, d\mu \geq s \cdot \mu(B)}$

Proof: Note that for all $n \in \mathbb{N}$ we have

$$\underbrace{\int_B f_n \, d\mu}_{=: I_n} = \int f_n 1_B \, d\mu \leq \underbrace{\int f_{n+1} 1_B \, d\mu}_{= I_{n+1}}$$

So $\{I_n\}$ is an increasing seq. in $\mathbb{R}^+$, so $\lim\limits_{n\to\infty} I_n$ exists in $\overline{\mathbb{R}}^+$.

If $s=0$ or $\mu(B)=0$, then the inequality above is obviously true.

We can therefore assume $s>0$ and $\mu(B)>0$.

Let $0<a<s$, $0<M<\mu(B)$.

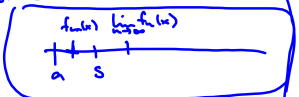
Set $A_n = \{x \in B : f_n(x) \geq a\} = B \cap f_n^{-1}([a,\infty)) \in \mathcal{A}$ for each n.

Since $f_n \leq f_{n+1}$ we get that $A_n \subseteq A_{n+1}$.

Moreover ... (to be finished next time).

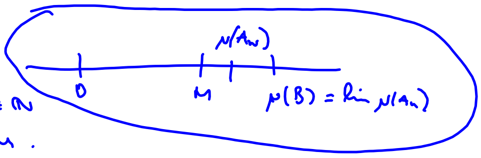
Added Feb. 14

$\bigcup\limits_{n=1}^{\infty} A_n = B$: Indeed, $\subseteq$ is obvious. Conversely, let $x \in B$. Since $\lim\limits_{n\to\infty} f_n(x) \geq s > a$, we can find some $m \in \mathbb{N}$ such that $f_m(x) \geq a$, i.e. $x \in A_m$.



We can now use "continuity from below" for μ: This gives that

$$\mu(B) = \lim_{n\to\infty} \mu(A_n)$$



We can then find some $N \in \mathbb{N}$ such that $\mu(A_N) > M$.

Then for each $n \geq N$ we get that

$$\int_B f_n \, d\mu = \int f_n \cdot 1_B \, d\mu \geq \int f_n \cdot 1_{A_n} \geq \int a \cdot 1_{A_n} \, d\mu = a\,\mu(A_n)$$

$\boxed{f_n \cdot 1_B \geq f_n \cdot 1_{A_n}}$ $\boxed{f_n \cdot 1_{A_n} \geq a \cdot 1_{A_n}}$

$$\geq a\,\mu(A_N) > a \cdot M$$

This implies that $\lim\limits_{n\to\infty} \int_B f_n \, d\mu \geq a \cdot M$ for all $0<a<s$ and all $0<M<\mu(B)$

which gives that $\lim\limits_{n\to\infty} \int f_n \, d\mu \geq s \cdot \mu(B)$.