7.4 (ctd.) $(X, \mathcal{A}, \mu)$ measure space.

Prop. Let $g \in \mathcal{S}^+$ and assume that $\{f_n\}$ is an increasing sequence in $\mathcal{S}^+$ such that $\lim_{n\to\infty} f_n(x) \geq g(x)$ for all $x \in X$.
Then $\lim_{n\to\infty} \int f_n \, d\mu \geq \int g \, d\mu$

Proof Write $g = \sum_{k=1}^{m} s_k 1_{B_k}$ (stand. form).

Let $k \in \{1, \dots, m\}$. Since $g(x) = s_k$ for $x \in B_k$, we have that $\lim_{n\to\infty} f_n(x) \geq g(x) = s_k$ for $x \in B_k$.

Applying the previous lemma, we get that

$$\lim_{n\to\infty} \int_{B_k} f_n \, d\mu \geq s_k \cdot \mu(B_k).$$

Now, since $X = \bigcup_{k=1}^{m} B_k$ and the B_k's are disjoint, $1_X = \sum_{k=1}^{m} 1_{B_k}$, so we get that

$$\int f_n \, d\mu = \int f_n \cdot \underbrace{\left(\sum_{k=1}^{m} 1_{B_k}\right)}_{1_X} d\mu = \int \left(\sum_{k=1}^{m} f_n 1_{B_k}\right) d\mu$$

$$= \sum_{k=1}^{m} \left(\int_{B_k} f_n \, d\mu\right) \quad \text{for every } n \in \mathbb{N}.$$

This gives that

$$\lim_{n\to\infty} \int f_n \, d\mu = \lim_{n\to\infty} \sum_{k=1}^{m} \left(\int_{B_k} f_n \, d\mu\right)$$

$$= \sum_{k=1}^{m} \left(\lim_{n\to\infty} \int_{B_k} f_n \, d\mu\right) \geq \underbrace{\sum_{k=1}^{m} s_k \, \mu(B_k)}_{= \int g \, d\mu}$$

as desired.

7.5 Integrals of nonnegative functions

$(X, \mathcal{A}, \mu)$ measure space.

Set $\overline{\mathcal{M}}^+ := \{ f : X \to \overline{\mathbb{R}}^+ \mid f \text{ is measurable} \}$

Def Let $f \in \overline{\mathcal{M}}^+$. We define the integral of f (w.r.t. μ) by

$$\boxed{\int f\,d\mu := \sup \underbrace{\left\{ \int g\,d\mu : g \in \mathcal{S}^+, g \le f \right\}}_{\subseteq \overline{\mathbb{R}}^+} \in \overline{\mathbb{R}}^+}$$

If $A \in \mathcal{A}$, then we set $\boxed{\int_A f\,d\mu := \int f \cdot 1_A\,d\mu}$

Note: If $f \in \mathcal{S}^+$, then this def. of $\int f\,d\mu$ agrees with the def. given in 7.4 (because we know that

$$\int g\,d\mu \le \int f\,d\mu \quad \text{for all } g \in \mathcal{S}^+ \text{ s.t. } g \le f).$$

Note: Let $f, h \in \overline{\mathcal{M}}^+$, $f \le h$. Then we have

- $\int f\,d\mu \le \int h\,d\mu$ → Let $g \in \mathcal{S}^+$, $g \le f$. Then $g \le h$, so $\int g\,d\mu \le \int h\,d\mu$ (by def. of $\int h\,d\mu$). Hence $\underbrace{\sup\{\int g\,d\mu : g \in \mathcal{S}^+, g \le f\}}_{= \int f\,d\mu} \le \int h\,d\mu$
- $\int_A f\,d\mu \le \int_A h\,d\mu$ for $A \in \mathcal{A}$. → because $f \cdot 1_A \le h \cdot 1_A$ and we can use the first part
- If $c \in \mathbb{R}^+$, then $\int cf\,d\mu = c \int d\mu$ → easy exercise

Prop. Let $f \in \overline{\mathcal{M}}^+$. Assume $\{h_n\}$ is an increasing sequence in $\mathcal{S}^+$ such that $h_n \to f$ pointwise on X. Then $\boxed{\int f\,d\mu = \lim_{n\to\infty} \int h_n\,d\mu}$

Proof: Since $\int h_n\,d\mu \le \int h_{n+1}\,d\mu$ (as $h_n \le h_{n+1}$) for all $n \in \mathbb{N}$, we know that $\lim_{n\to\infty} \int h_n\,d\mu$ exists in $\overline{\mathbb{R}}^+$. Moreover, as $h_n \le f$ and $h_n \in \mathcal{S}^+$ for all $n \in \mathbb{N}$, we get that $\int h_n\,d\mu \le \int f\,d\mu$ $\forall n$.

Taking the limit gives that $\lim_{n\to\infty} \int h_n\,d\mu \le \int f\,d\mu$.

On the other hand, if $g \in \mathcal{S}^+$, $g \le f$, then we have $\lim_{n\to\infty} h_n(x) = f(x) \ge g(x)$ for all $x \in X$, so we can apply the first proposition of today and get that

$$\lim_{n\to\infty} \int h_n\,d\mu \ge \int g\,d\mu. \quad \text{Hence,}$$

$$\lim_{n\to\infty} \int h_n\,d\mu \ge \underbrace{\sup\left\{\int g\,d\mu : g \in \mathcal{S}^+, g \le f\right\}}_{= \int f\,d\mu}$$

which is the reverse inequality.

Prop. (where the key idea of Lebesgue lies!)

Let $f \in \mathcal{M}^+$. Then there exists an increasing seq. $\{h_n\}$ in $\mathcal{S}^+$ such that $h_n \to f$ pointwise on X.

Moreover, for each $n \in \mathbb{N}$ and $x \in X$, we have that

$$0 \leq f(x) - h_n(x) < \frac{1}{2^n} \quad \text{when} \quad f(x) < 2^n$$

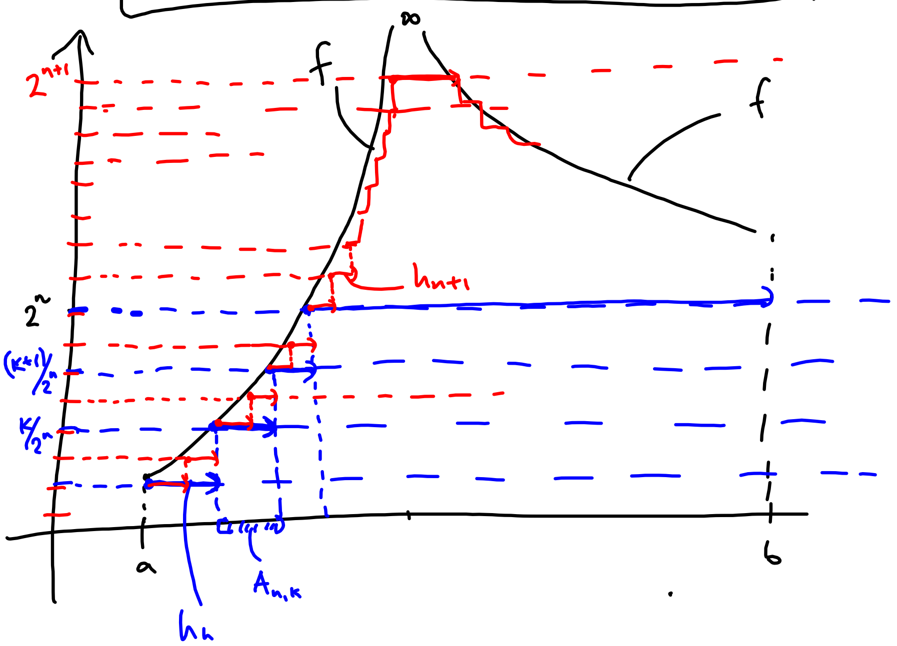
and $$h_n(x) = 2^n \quad \text{when} \quad f(x) \geq 2^n.$$

Proof Let $n \in \mathbb{N}$. We partition $[0, 2^n)$ in 2^{2n} subintervals of length $\frac{1}{2^n}$:

For $k = 0, 1, \dots, 2^{2n} - 1$, we set $I_{n,k} := \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$.

Then we set $A_{n,k} := f^{-1}(I_{n,k}) = \left\{x \in X : \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n}\right\}$ $\in \mathcal{A}$ since f is measurable

and define
$$\boxed{h_n := \left(\sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} 1_{A_{n,k}}\right) + 2^n \cdot 1_{f^{-1}([2^n, \infty])}}$$



Then one checks that $\{h_n\}$ does the job. F. ex.

if $f(x) < 2^n$, then $x \in A_{n,k}$ for some $0 \leq k \leq 2^{2n} - 1$,

so we get $h_n(x) = \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n} = h_n(x) + \frac{1}{2^n}$,

hence $0 \leq f(x) - h_n(x) < \frac{1}{2^n}$.