7.5 Int. of nonnegative functions II

$(X, \mathcal{A}, \mu)$ measure space.
$\overline{\mathcal{M}}^+ = \{ f : X \to \overline{\mathbb{R}}_+ : f \text{ is measurable} \}$

Combining the last two propositions, we get:

Corollary: Let $f \in \overline{\mathcal{M}}^+$. Then there exists a sequence $\{f_n\}$ in $\mathcal{S}^+$ which is increasing and converges pointwise to f on X
(We will just say that $\{f_n\}$ is a seq. in $\mathcal{S}^+$ s.t. $f_n \uparrow f$ ptwise on X),
and for every such sequence we have

$$\boxed{\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu}$$

We can now deduce the following:

Prop. Assume $f, g \in \overline{\mathcal{M}}^+$. Then $\boxed{\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu}$

Also, if $B \in \mathcal{A}$, then $\int_B (f+g) \, d\mu = \int_B f \, d\mu + \int_B g \, d\mu$.

Proof: Pick $\{f_n\}, \{g_n\}$ in $\mathcal{S}^+$ s.t. $f_n \uparrow f$, $g_n \uparrow g$ pt.wise on X.
Then $\{f_n + g_n\} \subseteq \mathcal{S}^+$ and $(f_n + g_n) \uparrow (f+g)$ — " —.
So the corollary above gives that

$$\int (f+g) \, d\mu = \lim_{n\to\infty} \int (f_n + g_n) \, d\mu \underset{\uparrow}{=} \lim_{n\to\infty} \left(\int f_n \, d\mu + \int g_n \, d\mu \right)$$

additivity of the integral of simple functions

$$= \lim_{n\to\infty} \int f_n \, d\mu + \lim_{n\to\infty} \int g_n \, d\mu = \int f \, d\mu + \int g \, d\mu.$$

If $B \in \mathcal{A}$, then we get

$$\int_B (f+g) \, d\mu = \int (f+g) 1_B \, d\mu = \int (f 1_B + g 1_B) \, d\mu$$

$$\underset{\uparrow}{=} \int f 1_B \, d\mu + \int g 1_B \, d\mu = \int_B f \, d\mu + \int_B g \, d\mu.$$

using the first part.

The Monotone Convergence Theorem (MCT): Let $\{f_n\}$ be a sequence in $\overline{\mathcal{M}}^+$ such that $f_n \nearrow f$ pointwise on X. Then $f \in \overline{\mathcal{M}}^+$ and

$$\boxed{\int f\, d\mu = \lim_{n\to\infty} \int f_n\, d\mu}$$

Also, if $B \in \mathcal{A}$, then $\int_B f\, d\mu = \lim_{n\to\infty} \int_B f_n\, d\mu$.

Proof. We have seen pointwise limits of meas. functions are measurable. So $f \in \overline{\mathcal{M}}^+$. Moreover, $\{\int f_n\, d\mu\}$ is an increasing sequence in $\overline{\mathbb{R}}^+$, so $\lim_{n\to\infty} \int f_n\, d\mu$ exists in $\overline{\mathbb{R}}^+$. As $f_n \le f$ on X we have $\int f_n\, d\mu \le \int f\, d\mu$ for all n, so taking the limit we get $\lim_{n\to\infty} \int f_n\, d\mu \le \int f\, d\mu$.

Conversely, for each $n \in \mathbb{N}$, we can pick a sequence $\{h_{n,m}\}_{m\in\mathbb{N}}$ in $\mathcal{S}^+$ s.t. $h_{n,m} \nearrow f_n$ ptwise on X (as in the prop. with Lebesgue's key idea). Set $g_n = h_{n,n} \in \mathcal{S}^+$ for each $n \in \mathbb{N}$. Then, "by construction", ~~the~~ $\{g_n\}$ is increasing and $g_n \to f$ ptwise on X.

(check this) Since $g_n \le f_n$ for all $n \in \mathbb{N}$, we get that

$$\lim_{n\to\infty} \int f_n\, d\mu \ge \lim_{n\to\infty} \int g_n\, d\mu = \int f\, d\mu$$

using the prop. involving functions in $\mathcal{S}^+$.

showing the reverse inequality.

If $B \in \mathcal{A}$, then $f_n 1_B \nearrow f 1_B$ ptwise on X, so we get

$$\lim_{n\to\infty} \underbrace{\int f_n 1_B\, d\mu}_{\int_B f_n\, d\mu} = \underbrace{\int f 1_B\, d\mu}_{= \int_B f\, d\mu}.$$

Corollary Assume $\{f_k\}$ is a seq. in $\overline{\mathcal{M}}^+$ and $B \in \mathcal{A}$.

Then $\boxed{\int_B \left(\sum_{k=1}^{\infty} f_k\right) d\mu = \sum_{k=1}^{\infty} \left(\int_B f_k\, d\mu\right)}$

Proof Set $g_n := \sum_{k=1}^{n} f_k \in \overline{\mathcal{M}}^+$ for each $n \in \mathbb{N}$.

Then $\{g_n\}$ is increasing and $\lim_{n\to\infty} g_n = \sum_{k=1}^{\infty} f_k$ (by def.).

So we get that

$$\int_B \left(\sum_{k=1}^{\infty} f_k\right) d\mu = \int_B \left(\lim_{n\to\infty} g_n\right) d\mu \underset{\text{MCT}}{=} \lim_{n\to\infty} \int_B g_n\, d\mu$$

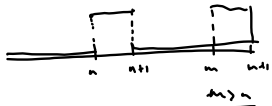
$$= \lim_{n\to\infty} \int_B \left(\sum_{k=1}^{n} f_k\right) d\mu \underset{\text{additivity of the int.}}{=} \lim_{n\to\infty} \sum_{k=1}^{n} \left(\int_B f_k\, d\mu\right)$$

$$= \sum_{k=1}^{\infty} \left(\int_B f_k\, d\mu\right).$$

Note: There are sequences $\{f_n\}$ in $\overline{\mathcal{M}}^+$ converging ptwise on X to some $f \in \overline{\mathcal{M}}^+$ such that $\int f\,d\mu \neq \lim_{n\to\infty} \int f_n\,d\mu$.

For example, let μ be the Lebesgue measure on $\mathbb{R}$.

For $n \in \mathbb{N}$, set $f_n := \mathbb{1}_{[n,n+1]} \in \overline{\mathcal{M}}^+$



Then $f_n \to 0$ ptwise on $\mathbb{R}$, Moreover

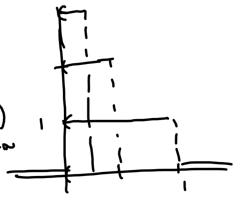
$\int f_n\,d\mu = \mu([n,n+1]) = 1$ for all n, so $\lim_{n\to\infty} \int f_n\,d\mu = \underline{\underline{1}}$,

while $\int (\lim_{n\to\infty} f_n)\,d\mu = \int 0\,d\mu = \underline{\underline{0}}$

Or we can set $g_n = n \cdot \mathbb{1}_{(0,\frac{1}{n}]}$ $\forall n \in \mathbb{N}$.

Then $g_n \to 0$ ptwise on $\mathbb{R}$, $\int g_n\,d\mu = n\,\mu((0,\frac{1}{n}]) = 1$ $\forall n \in \mathbb{N}$

while $\int (\lim_{n\to\infty} g_n)\,d\mu = 0$



The "best" one can say in general is:

Fatou's lemma Let $\{f_n\} \subseteq \overline{\mathcal{M}}^+$. Then

$$\boxed{\int \left(\liminf_{n\to\infty} f_n\right) d\mu \leq \liminf_{n\to\infty} \int f_n\,d\mu}$$

Sketch of the proof: For $k \in \mathbb{N}$, set $g_k(x) := \inf\{f_n(x) : n \geq k\}$, $x \in X$.

Then $\{g_k\} \subseteq \overline{\mathcal{M}}^+$, $g_k \nearrow \liminf_{n\to\infty} f_n$ ptwise on X.

As $g_k \leq f_n$ for $n \geq k$, we have $\int g_k\,d\mu \leq \int f_n\,d\mu$ when $n \geq k$.

So $\int g_k\,d\mu \leq \inf\{\int f_n\,d\mu : n \geq k\} \leq \liminf_{n\to\infty} \int f_n\,d\mu$ $\forall k \in \mathbb{N}$.

Hence we get that

$$\int \left(\liminf_{n\to\infty} f_n\right) d\mu = \lim_{k\to\infty} \int g_k\,d\mu \leq \liminf_{n\to\infty} \int f_n\,d\mu.$$

by the MCT

Some other useful properties of the integral: Let $f, g \in \overline{\mathcal{M}}^+$.

- $\boxed{f = 0 \text{ a.e.} \Leftrightarrow \int f\,d\mu = 0}$

  It follows that if $E \in \mathcal{A}$ and $\mu(E) = 0$, then $\int_E f\,d\mu = 0$ (because $f\mathbb{1}_E = 0$ a.e.)

- $\boxed{f = g \text{ a.e.} \Rightarrow \int f\,d\mu = \int g\,d\mu}$.

- If $\underbrace{\int f\,d\mu < \infty}$, then f is finite a.e.

  "f is integrable over X"

Exercises

- If $A, B \in \mathcal{A}$, $A \cap B = \emptyset$, then

  $$\int_{A\cup B} f\,d\mu = \int_A f\,d\mu + \int_B f\,d\mu.$$

  More generally, if $\nu : \mathcal{A} \to \overline{\mathbb{R}}^+$ is defined by $\boxed{\nu(A) := \int_A f\,d\mu,\ A \in \mathcal{A}}$

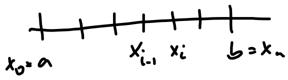
  then ν is a measure on $(X, \mathcal{A})$.

Riemann vs Lebesgue Let us consider the σ-algebra $\mathcal{A}$ of Lebesgue measurable subsets of $[a,b]$, $a,b\in\mathbb{R}$, $a<b$, and the restriction μ of the Leb. measure on $\mathbb{R}$ to $\mathcal{A}$.

Let $f:[a,b]\to[0,\infty)$ be a bounded function, & and assume that f is Riemann-integrable. Then f is Lebesgue-integrable and

$$\boxed{\int_{[a,b]} f\,d\mu = \int_a^b f(x)\,dx}$$

A brief sketch: Let $\mathcal{P}=\{x_0,x_1,\dots,x_n\}$ be a partition of $[a,b]$.



Set $\begin{cases} m_i = \inf\{f(x) : x\in(x_{i-1},x_i]\} \\ M_i = \sup\{f(x) : \text{—''—}\} \end{cases}$

and $\begin{cases} L(\mathcal{P}) = \sum_{i=1}^n m_i(x_i-x_{i-1}) \\ U(\mathcal{P}) = \sum_{i=1}^n M_i(x_i-x_{i-1}) \end{cases}$

Recall that $\underline{\int_a^b} f := \sup\{L(\mathcal{P}) : \mathcal{P} \text{ part.}\}$,

$\overline{\int_a^b} f := \inf\{U(\mathcal{P}) : \mathcal{P} \text{ part.}\}$, and

$\underline{\int_a^b} f \le \overline{\int_a^b} f$ with equality precisely when f is Riemann-integrable.

Can also define $\Phi_{\mathcal{P}} = \sum_{i=1}^n m_i \mathbb{1}_{(x_{i-1},x_i]}$, $\Psi_{\mathcal{P}} = \sum_{i=1}^n M_i \mathbb{1}_{(x_{i-1},x_i]}$ which are both in $\mathcal{S}^+$, and $\Phi_{\mathcal{P}} \le f \le \Psi_{\mathcal{P}}$.

Clearly, $L(\mathcal{P}) = \int \Phi_{\mathcal{P}}\,d\mu$ and $U(\mathcal{P}) = \int \Psi_{\mathcal{P}}\,d\mu$.

Taking sup over all $\mathcal{P}$ we get

$$\underline{\int_a^b} f \le \sup\left\{\int \Phi_{\mathcal{P}}\,d\mu \mid \mathcal{P} \text{ part.}\right\}$$
$$\le \underbrace{\sup\left\{\int g\,d\mu : g\in\mathcal{S}^+,\ g\le f\right\}}_{=:\ \underline{I}(f)}$$

Define similarly $\overline{I}(f) := \inf\left\{\int h\,d\mu : h\in\mathcal{S}^+,\ f\le h\right\}$.

Then $\underline{\int_a^b} f \le \underline{I}(f) \le \overline{I}(f) \le \overline{\int_a^b} f$

= since f is Riem. int.

Now f is Lebesgue-measurable (!!), so $\underline{I}(f) = \int f\,d\mu$, and we get that $\int f\,d\mu = \int_a^b f(x)\,dx$

read yourself!