7.6 (and a bit from 7.9): Integrable functions

$(X, \mathcal{A}, \mu)$ measure space.

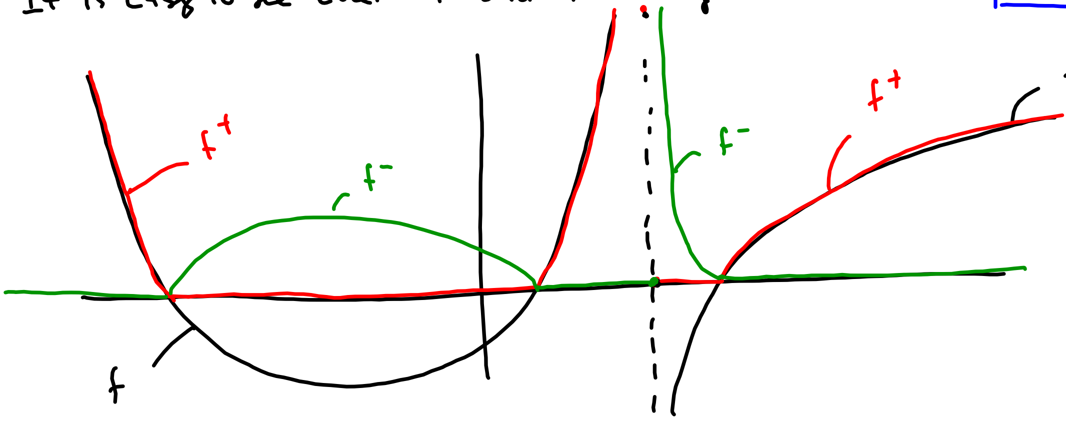
$\overline{\mathcal{M}} = \{ f : X \to \overline{\mathbb{R}} : f \text{ is measurable} \}$

$\mathcal{M} = \{ f : X \to \mathbb{R} : \text{ — '' — } \}$

$[\max(f,g)](x) := \max(f(x), g(x))$ for all $x \in X$. Similarly for $\min(f,g)$

For $f \in \overline{\mathcal{M}}$, set $\begin{cases} f^+ := \max(f, 0) \\ f^- := -\min(f,0) = \max(-f, 0) \end{cases}$

It is easy to see that f^+ and f^- belong to $\overline{\mathcal{M}}^+$.



We then have that $\boxed{\begin{array}{l} f = f^+ - f^- \\ |f| = f^+ + f^- \end{array}}$

We say that $f \in \overline{\mathcal{M}}$ is <u>integrable</u> (w.r.t. μ) when $\int f^+ \, d\mu$ and $\int f^- \, d\mu$ both are <u>finite</u> numbers, in which case we set

$$\boxed{\int f \, d\mu := \int f^+ \, d\mu - \int f^- \, d\mu}$$

If $A \in \mathcal{A}$, we also set $\int_A f \, d\mu := \int f 1_A \, d\mu = \int_A f^+ \, d\mu - \int_A f^- \, d\mu$

Note: Let $f \in \overline{\mathcal{M}}$. Then $\boxed{f \text{ is integrable} \iff \int |f| \, d\mu < \infty}$

$\iff |f|$ is integrable.

(The reason is that $\int |f| \, d\mu = \int (f^+ + f^-) \, d\mu = \int f^+ \, d\mu + \int f^- \, d\mu$)

Note: Assume $f: X \to \overline{\mathbb{R}}$ is integrable.
Then $\int f^+ d\mu < \infty$ and $\int f^- d\mu < \infty$, so we get that f^+ and f^- are both finite almost everywhere (cf. Exercise). We may therefore redefine f on a set of measure zero, so that f is finite everywhere and its integral still has the same value. We will therefore mostly deal with the space

$$\boxed{\underbrace{\mathcal{L}^1_{\mathbb{R}}(X, \mathcal{A}, \mu)}_{\mathcal{L}^1_{\mathbb{R}}} := \{ f \in \mathcal{M} : f \text{ is integrable (w.r.t. } \mu) \} = \left\{ f: X \to \mathbb{R} : \begin{array}{l} f \text{ is measurable and} \\ \int |f| \, d\mu < \infty \end{array} \right\}}$$

We can also consider ~~the~~ complex functions:
Setting $\overline{\mathbb{C}} := \{ x + iy : x, y \in \overline{\mathbb{R}} \}$.

$\boxed{\begin{array}{l} \mathrm{Re}(x+iy) := x \in \overline{\mathbb{R}} \\ \mathrm{Im}(x+iy) := y \in \overline{\mathbb{R}} \end{array}}$

We say that $f: X \to \overline{\mathbb{C}}$ is measurable when $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are measurable as functions from X to $\overline{\mathbb{R}}$. Moreover, we say that it is integrable (w.r.t. μ) when $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are both integrable, in which case we set $\boxed{\int f \, d\mu := \int (\mathrm{Re} f) \, d\mu + i \int (\mathrm{Im} f) \, d\mu}$

Note: Assume $f: X \to \overline{\mathbb{C}}$ is measurable.
$\boxed{\text{Then } f \text{ is integrable} \iff \int |f| \, d\mu < \infty}$

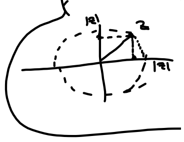
Proof: ($\Rightarrow$) Assume that $\mathrm{Re}(f)$, $\mathrm{Im}(f)$ are both integrable.
So (by the previous note) $\int |\mathrm{Re}(f)| \, d\mu < \infty$, $\int |\mathrm{Im}(f)| \, d\mu < \infty$

As $|f| = |\mathrm{Re}(f) + i \, \mathrm{Im}(f)| \underset{\Delta\text{-inequality}}{\leq} |\mathrm{Re}(f)| + |\mathrm{Im}(f)|$,

we get
$$\int |f| \, d\mu \leq \int \big(|\mathrm{Re}(f)| + |\mathrm{Im}(f)| \big) \, d\mu = \int |\mathrm{Re}(f)| \, d\mu + \int |\mathrm{Im}(f)| \, d\mu < \infty.$$

($\Leftarrow$) Assume $\int |f| \, d\mu < \infty$.
Since $|\mathrm{Re}(f)| \leq |f|$ and $|\mathrm{Im}(f)| \leq |f|$

$|\mathrm{Re}\, z| \leq |z|$
$|\mathrm{Im}\, z| \leq |z|$



we get that
$$\begin{cases} \int |\mathrm{Re}(f)| \, d\mu \leq \int |f| \, d\mu < \infty \\ \int |\mathrm{Im}(f)| \, d\mu \quad \text{—''—} \end{cases}$$

which gives that $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are both integrable.

Note: If $f: X \to \overline{\mathbb{C}}$ is integrable, then $(\mathrm{Re} f)^+$, $(\mathrm{Re} f)^-$, $(\mathrm{Im} f)^+$, $(\mathrm{Im} f)^-$ are all finite almost everywhere.
This means that we can redefine f (if necessary) and assume that it takes values in $\mathbb{C}$, without changing the value of $\int f \, d\mu$.

We will therefore mostly consider

$$\underbrace{\mathcal{L}^1_{\mathbb{C}}(X, \mathcal{A}, \mu)}_{\mathcal{L}^1_{\mathbb{C}}} := \{ f: X \to \mathbb{C} \mid f \text{ is integrable} \} = \left\{ \text{—''—} \,\middle|\, \begin{array}{l} f \text{ is measurable,} \\ \int |f| \, d\mu < \infty \end{array} \right\}$$

Note that $\mathcal{L}^1_{\mathbb{R}} \subseteq \mathcal{L}^1_{\mathbb{C}}$ (But, as we will see, $\mathcal{L}^1_{\mathbb{R}}$ is a real vector space, while $\mathcal{L}^1_{\mathbb{C}}$ is a complex vector space)

Notation: $\mathbb{F} := \mathbb{R}$ or $\mathbb{C}$

Proposition Consider the vector space $\mathcal{F}(X,\mathbb{F})$ over $\mathbb{F}$ consisting of all functions from X to $\mathbb{F}$ (equipped with its natural pointwise operations). Then $\mathcal{L}^1_{\mathbb{F}}$ is a subspace of $\mathcal{F}(X,\mathbb{F})$.
Moreover, the map $f \longmapsto \int f\,d\mu$ from $\mathcal{L}^1_{\mathbb{F}}$ to $\mathbb{F}$ is linear. (In fact, $f \longmapsto \int_A f\,d\mu$, where $A \in \mathcal{A}$, is also linear.)

Proof Let $f, g \in \mathcal{L}^1_{\mathbb{F}}$ and $\alpha \in \mathbb{F}$. Then, since $|f+g| \le |f|+|g|$ and $|\alpha f| = |\alpha|\,|f|$, we get that $\int |f+g|\,d\mu < \infty$ and $\int |\alpha f|\,d\mu < \infty$, i.e. $\begin{cases} f+g \in \mathcal{L}^1_{\mathbb{F}} \\ \alpha f \in \mathcal{L}^1_{\mathbb{F}} \end{cases}$

So $\mathcal{L}^1_{\mathbb{F}}$ is a subspace of $\mathcal{F}(X,\mathbb{F})$.

Assume $f, g \in \mathcal{L}^1_{\mathbb{R}}$ and set $h = f+g \in \mathcal{L}^1_{\mathbb{R}}$.

Then $h^+ - h^- = h = f+g = (f^+ - f^-) + (g^+ - g^-)$

So $h^+ + f^- + g^- = f^+ + g^+ + h^-$

By integrating to μ, we get that

$$\int h^+\,d\mu + \int f^-\,d\mu + \int g^-\,d\mu = \int f^+\,d\mu + \int g^+\,d\mu + \int h^-\,d\mu$$

(and all integrals are finite). Thus we obtain

$$\underbrace{\int h^+\,d\mu - \int h^-\,d\mu}_{=\int h\,d\mu} = \underbrace{\int f^+\,d\mu - \int f^-\,d\mu}_{=\int f\,d\mu} + \underbrace{\int g^+\,d\mu - \int g^-\,d\mu}_{=\int g\,d\mu}$$

as desired.

Assume now $f, g \in \mathcal{L}^1_{\mathbb{C}}$.

Then $f+g = \underbrace{\mathrm{Re}(f+g)}_{=\mathrm{Re}(f)+\mathrm{Re}(g)} + i\,\underbrace{\mathrm{Im}(f+g)}_{=\mathrm{Im}(f)+\mathrm{Im}(g)}$

$$\begin{aligned} \text{So } \int (f+g)\,d\mu &= \int (\mathrm{Re}(f)+\mathrm{Re}(g))\,d\mu + i\int (\mathrm{Im}(f)+\mathrm{Im}(g))\,d\mu \\ &= \int \mathrm{Re}(f)\,d\mu + \int \mathrm{Re}(g)\,d\mu + i\left(\int \mathrm{Im}(f)\,d\mu + \int \mathrm{Im}(g)\,d\mu\right) \\ &= \underbrace{\int \mathrm{Re}(f)\,d\mu + i\int \mathrm{Im}(f)\,d\mu}_{=\int f\,d\mu} + \underbrace{\int \mathrm{Re}(g)\,d\mu + i\int \mathrm{Im}(g)\,d\mu}_{=\int g\,d\mu}. \end{aligned}$$

by the proof in the $\mathcal{L}^1_{\mathbb{R}}$-case

- We leave the proof of $\int \alpha f\,d\mu = \alpha \int f\,d\mu$ as an exercise.

- If $A \in \mathcal{A}$, we get

$\alpha, \beta \in \mathbb{F}$

$$\begin{aligned} \int_A (\alpha f + \beta g)\,d\mu &= \int (\alpha f + \beta g)1_A\,d\mu \\ &= \int (\alpha f 1_A + \beta g 1_A)\,d\mu \\ &= \int \alpha f 1_A\,d\mu + \int \beta g 1_A\,d\mu \\ &= \alpha \underbrace{\int f 1_A\,d\mu}_{=\int_A f\,d\mu} + \beta \underbrace{\int g 1_A\,d\mu}_{=\int_A g\,d\mu}. \end{aligned}$$

Proposition

1) Let $f: X \to \mathbb{F}$ be measurable. Then $\boxed{\int |f|\, d\mu = 0 \iff f = 0 \text{ a.e.}}$

2) Let $f \in \mathcal{L}^1_{\mathbb{F}}$, $A \in \mathcal{A}$, $\mu(A) = 0$.
Then $\int_A f\, d\mu = 0$

3) Assume $f, g \in \mathcal{L}^1_{\mathbb{F}}$, $f = g$ a.e.
Then $\int f\, d\mu = \int g\, d\mu$.

Proof 1) Since $|f| \geq 0$, we know that $\int |f|\, d\mu = 0 \iff |f| = 0$ a.e. $\iff f = 0$ a.e.

(cf. Exercise)

2) We may assume $\mathbb{F} = \mathbb{C}$. We then know (from an exercise!) that

$$\int_A \mathrm{Re}(f)^{\pm}\, d\mu = 0 \quad \text{and} \quad \int_A \mathrm{Im}(f)^{\pm}\, d\mu = 0$$

Hence

$$\int_A f\, d\mu = \int_A \mathrm{Re}(f)^{+}\, d\mu - \int_A \mathrm{Re}(f)^{-}\, d\mu + i\left(\int_A \mathrm{Im}(f)^{+}\, d\mu - \int_A \mathrm{Im}(f)^{-}\, d\mu\right) = 0$$

3) Let $B = \{x \in X \mid f(x) = g(x)\}$. Then $\mu(B^c) = 0$ by assumption. So we get

$$\int f\, d\mu = \int f \cdot (1_B + 1_{B^c})\, d\mu = \int_B f\, d\mu + \underbrace{\int_{B^c} f\, d\mu}_{\substack{= 0 \\ \text{by using 2)}}}$$

$$= \int_B g\, d\mu = \cdots = \int g\, d\mu$$

$f = g$ on B

Another useful property:

Assume $f, g \in \mathcal{L}^1_{\mathbb{R}}$ and $f \leq g$ on X.
Then $\int f\, d\mu \leq \int g\, d\mu$.

Proof: Since $g - f \geq 0$ on X, we get that

$$\int g\, d\mu = \int (f + (g - f))\, d\mu = \int f\, d\mu + \underbrace{\int (g - f)\, d\mu}_{\geq 0}$$

$$\geq \int f\, d\mu.$$