7.6 (and 7.9) Integrable functions II

$(X, \mathcal{A}, \mu)$ measure space. $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

$$\underbrace{\mathcal{L}^1_{\mathbb{F}}(X, \mathcal{A}, \mu)}_{\mathcal{L}^1_{\mathbb{F}}} = \{ f : X \to \mathbb{F} \mid \underbrace{f \text{ is measurable and } \int |f| \, d\mu < \infty}_{f \text{ is \underline{integrable} (w.r.t. } \mu)} \}$$

Another useful property!

Prop $\boxed{\text{Let } f \in \mathcal{L}^1_{\mathbb{F}}. \text{ Then } \left| \int f \, d\mu \right| \leq \int |f| \, d\mu}$

Proof When $\mathbb{F} = \mathbb{R}$, this is easy. So let's assume $\mathbb{F} = \mathbb{C}$.

Set $z = \int f \, d\mu \in \mathbb{C}$. Write $z = |z| e^{i\theta}$ for some $\theta \in [0, 2\pi)$.

polar decomposition

Then

$$\underbrace{|z|}_{\in \mathbb{R}} = e^{-i\theta} z = e^{-i\theta} \int f \, d\mu = \int e^{-i\theta} f \, d\mu$$

$$= \underbrace{\int \mathrm{Re}(e^{-i\theta} f) \, d\mu}_{\in \mathbb{R}} + i \underbrace{\int \mathrm{Im}(e^{-i\theta} f) \, d\mu}_{\in \mathbb{R}}$$

So $\int \mathrm{Im}(e^{-i\theta} f) \, d\mu = 0$, and we get

$$\left| \int f \, d\mu \right| = |z| = \int \mathrm{Re}(e^{-i\theta} f) \, d\mu \leq \int |f| \, d\mu$$

since $\mathrm{Re}(e^{-i\theta} f) \leq |e^{-i\theta} f| = |f|$

where we use the previous result.

Lebesgue's dominated convergence theorem (LDCT):

Let f_n and f be complex (or real-valued) measurable functions ($n \in \mathbb{N}$).

Assume

i) $\lim_{n\to\infty} f_n = f$ pointwise a.e. on X.

ii) there exists $g: X \to [0,\infty]$ measurable such that $|f_n| \leq g$ a.e. on X and $\int g \, d\mu < \infty$ (i.e. g is integrable).

Then all f_n's and f are integrable (w.r.t. μ). Moreover, we have that $\boxed{\lim_{n\to\infty} \int |f_n - f| \, d\mu = 0}$

and $\boxed{\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu}$

Proof: For simplicity we assume that $\lim_{n\to\infty} f_n = f$ and $|f_n| \leq g$ hold everywhere on X (and leave the general case as an exercise).

Since $|f_n| \leq g$ $\forall n$, we get that $\int |f_n| \, d\mu \leq \int g \, d\mu < \infty$ $\forall n \in \mathbb{N}$.
So f_n is int. (w.r.t. μ) $\forall n \in \mathbb{N}$.

Moreover, we have $|f| = \underbrace{\lim_{n\to\infty} |f_n|}_{\leq g}$ ptwise on X, so we also get that $\int |f| \, d\mu \leq \int g \, d\mu < \infty$, so f is int. (w.r.t. μ).

Since $|f_n - f| \leq |f_n| + |f| \leq 2g$ for all n, we have $2g - |f_n - f| \geq 0$. As $\lim_{n\to\infty} |f_n - f| = 0$, we get

$$\int 2g \, d\mu = \int \lim_{n\to\infty} (2g - |f_n - f|) \, d\mu \leq \liminf_{n\to\infty} \int (2g - |f_n - f|) \, d\mu$$

Fatou's lemma

i.e. $\cancel{\int 2g \, d\mu} \leq \cancel{\int 2g \, d\mu} + \underbrace{\liminf_{n\to\infty} \left(-\int |f_n - f| \, d\mu\right)}_{= -\limsup_{n\to\infty} \int |f_n - f| \, d\mu}$

which gives that $\limsup_{n\to\infty} \int |f_n - f| \, d\mu \leq 0$, which implies that $\lim_{n\to\infty} \int |f_n - f| \, d\mu = 0$. Finally, we obtain that

$$\left| \int f_n \, d\mu - \int f \, d\mu \right| = \left| \int (f_n - f) \, d\mu \right| \leq \underbrace{\int |f_n - f| \, d\mu}_{\to 0 \text{ as } n\to\infty}$$

by the previous prop.

So $\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$.

<u>Example</u> Consider the Leb. measure μ on the Borel σ-alg. on $[0,1]$.

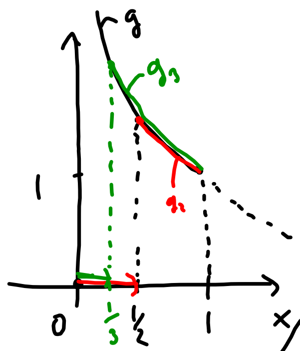
Let $f_n(x) = \dfrac{n \sin(x)}{1+n^2\sqrt{x}}$, $0 \le x \le 1$.

Then each f_n is continuous, so it is measurable. Moreover,

$f_n(x) \to 0$ for every $x \in [0,1]$, i.e. <u>$f_n \to 0$ ptwise on $[0,1]$.</u>

Let $x \neq 0$. Then $|f_n(x)| \le \dfrac{n}{1+n^2\sqrt{x}} \le \dfrac{n}{n^2\sqrt{x}} = \dfrac{1}{n\sqrt{x}} \le \dfrac{1}{\sqrt{x}}$

Set $g(x) := \begin{cases} 1/\sqrt{x}, & \text{if } 0<x\le 1 \\ 0, & \text{if } x=0 \end{cases}$. Then <u>$|f_n| \le g$ on $[0,1]$.</u>



Note that g is <u>integrable</u>: For each $m \in \mathbb{N}$,

Set $g_m(x) = \begin{cases} 1/\sqrt{x}, & \frac{1}{m} \le x \le 1 \\ 0, & 0 \le x < \frac{1}{m} \end{cases}$ <u>measurable</u>

Then $g_m \nearrow g$ ptwise on $[0,1]$, so the

MCT gives that

$$\int_{[0,1]} g\, d\mu = \lim_{m\to\infty} \int_{[0,1]} g_m\, d\mu = \lim_{m\to\infty} \int_{[1/m,1]} g_m\, d\mu$$

$$= \lim_{m\to\infty} \int_{1/m}^{1} \frac{1}{\sqrt{x}}\, dx = \lim_{m\to\infty} \Big[2\sqrt{x}\Big]_{x=1/m}^{x=1}$$

g_m is continuous on $[1/m,1]$, so it is Riemann-integrable.

$$= \lim_{m\to\infty} 2 - 2\sqrt{1/m} = 2 < \infty.$$

The LDCT gives that $\displaystyle\lim_{n\to\infty} \int_{[0,1]} f_n\, d\mu = \int_{[0,1]} \underbrace{\lim_{n\to\infty} f_n}_{=0}\, d\mu = \underline{\underline{0}}$

<u>Note</u>: In the LDCT, it is important to check that all the assumptions hold.

F.ex. <u>$f_n = n\,\mathbb{1}_{(0,1/n)}$</u> (or $f_n = \frac{1}{n}\mathbb{1}_{[0,n]}$)

μ = Leb. measure on $\mathcal{B}_{\mathbb{R}}$. Then $\displaystyle\int_{\mathbb{R}} f_n\, d\mu = n \cdot \frac{1}{n} = 1 \quad \forall n \in \mathbb{N}$

But $\displaystyle\int_{\mathbb{R}} \lim_{n\to\infty} f_n\, d\mu = \int_{\mathbb{R}} 0\, d\mu = 0$

An application of the LDCT.

Let I be some interval, $I \subseteq \mathbb{R}$. $(X, \mathcal{A}, \mu)$ measure space.

Consider a function $u : I \times X \to \mathbb{F}$
$$(t,x) \longmapsto u(t,x)$$

- Assume that the function $x \longmapsto u(t,x)$ is measurable (w.r.t. $\mathcal{A}$) for each $t \in I$.
- Moreover assume there exists $g : X \to [0,\infty)$ which is integrable (w.r.t. μ) and satisfies that $\overline{|u(t,x)| \leq g(x)}$ for all $(t,x) \in I \times X$.

Then $\boxed{x \longmapsto u(t,x) \text{ is integrable for all } t \in I}$.

- Assume further that $t_0 \in I$ and $\boxed{\lim_{t \to t_0} u(t,x) = u(t_0,x) \quad \forall x \in X}$

(i.e. $t \longmapsto u(t,x)$ is cont. at t_0 for each $x \in X$)

Then
$$\boxed{\lim_{t \to t_0} \underbrace{\int u(t,x)\, d\mu(x)}_{=: v(t)} = \underbrace{\int \underbrace{\lim_{t \to t_0} u(t,x)}_{= u(t_0,x)}\, d\mu(x)}_{= v(t_0)}}$$

i.e. v is cont. at t_0

The idea of the proof is as follows:

Consider any sequence $\{t_n\} \subseteq I$ such that $t_n \to t_0$

Define $u_n(x) := u(t_n, x)$, $x \in X$. Then $|u_n| \leq g$ for all n

(and $\lim_{n \to \infty} u_n(x) = u(t_0, x)$ for all if the last assumption holds).

Then LDCT gives that all the u_n are integrable (w.r.t. μ), and

$$\lim_{n \to \infty} \underbrace{\int u_n(x)\, d\mu(x)}_{= \underbrace{\int u(t_n,x)\, d\mu(x)}_{= v(t_n)}} = \int \lim_{n\to\infty} u_n(x)\, d\mu(x) = \underbrace{\int u(t_0,x)\, d\mu(x)}_{v(t_0)}$$

when the last assumption holds.

Since this holds for every seq. $\{t_n\}$ converging to t_0, this implies that $v(t) \to v(t_0)$ as $t \to t_0$.