8.1 and 8.2 : Outer measures and measurable sets.

Let X be a nonempty set, and let $\mathcal{R}$ be a family of subsets of X. Assume $\rho: \mathcal{R} \to [0,\infty]$ is a function.

Want to find conditions on $\mathcal{R}$ and ρ such that we can extend ρ to a measure on a σ-algebra containing $\mathcal{R}$ (hence containing $\Sigma(\mathcal{R})$ ← the σ-alg. generated by $\mathcal{R}$)

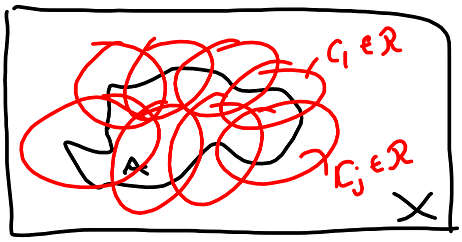
To start with, we only assume:

(i) $\emptyset \in \mathcal{R}$
(ii) $X = \bigcup_{n=1}^{\infty} R_n$ where $R_n \in \mathcal{R}$ for each $n \in \mathbb{N}$.
(iii) $\rho(\emptyset) = 0$

Note: If $\mathcal{R}$ contains $\emptyset$ and X, then i) and ii) are satisfied.

We can then define $\mu^*(A) = \inf \left\{ \underbrace{\sum_{n=1}^{\infty} \rho(C_n)}_{\in [0,\infty]} \mid \{C_n\}_{n\in\mathbb{N}} \subseteq \mathcal{R} \text{ and } A \subseteq \bigcup_{n=1}^{\infty} C_n \right\}$ for every $A \subseteq X$.

$\{C_n\}$ is a "$\mathcal{R}$-covering" of A



Note that this makes sense because $A \subseteq X = \bigcup_{n=1}^{\infty} R_n$ where $\{R_n\}$ is as in (ii)

In general, μ^* is not a measure on $\mathcal{P}(X)$. However, it is an **outer measure** (on $\mathcal{P}(X)$), that is, it satisfies the foll. conditions:

(OM1): $\mu^*(\emptyset) = 0$
(OM2): $A \subseteq B \subseteq X \Rightarrow \mu^*(A) \leq \mu^*(B)$
(OM3): If $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}(X)$, then $\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$

Indeed, let's check this.

(OM1) Let $C_n := \emptyset$ for all $n \in \mathbb{N}$. Then $\emptyset \subseteq \bigcup_{n=1}^{\infty} C_n$ and $\sum_{n=1}^{\infty} \underbrace{\rho(C_n)}_{=0} = 0$.
$\emptyset \in \mathcal{R}$

This implies that $\mu^*(\emptyset) = 0$.

(OM2) Let $A \subseteq B \subseteq X$. Let $\{D_n\}_{n\in\mathbb{N}}$ be a $\mathcal{R}$-covering of B.

Then $A \subseteq \bigcup_{n\in\mathbb{N}} D_n$, so $\{D_n\}$ is also a $\mathcal{R}$-covering of A.

Hence $\mu^*(A) \leq \sum_{n=1}^{\infty} \rho(D_n)$.

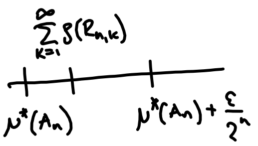
Taking the inf over all $\mathcal{R}$-coverings of B, we get that

$$\mu^*(A) \leq \mu^*(B).$$

(OM3) Consider $A = \bigcup_{n\in\mathbb{N}} A_n$ where $A_n \subseteq X$, $n \in \mathbb{N}$.

If $\mu^*(A_m) = \infty$ for some $m \in \mathbb{N}$, then (OM3) is obviously true.

So we may assume that $\mu^*(A_n) < \infty$ for every $n \in \mathbb{N}$.



Let $\varepsilon > 0$. For each $n \in \mathbb{N}$, we can then find a seq. $\{R_{n,k}\}_{k\in\mathbb{N}}$ in $\mathcal{R}$ such that

$$A_n \subseteq \bigcup_{k=1}^{\infty} R_{n,k} \quad \text{and} \quad \sum_{k=1}^{\infty} \rho(R_{n,k}) < \mu^*(A_n) + \frac{\varepsilon}{2^n}$$

Now, $\{R_{n,k}\}_{(n,k)\in\mathbb{N}\times\mathbb{N}}$ is a countable family of sets in $\mathcal{R}$,

so we may list its elements as a sequence $\{B_m\}_{m\in\mathbb{N}}$.

Then we have that

$$A = \bigcup_{n\in\mathbb{N}} A_n \subseteq \bigcup_{n\in\mathbb{N}} \left(\bigcup_{k=1}^{\infty} R_{n,k} \right) = \bigcup_{(n,k)\in\mathbb{N}\times\mathbb{N}} R_{n,k} = \bigcup_{m\in\mathbb{N}} B_m$$

So we get that $\{B_m\}_{m\in\mathbb{N}}$ is a $\mathcal{R}$-covering of A. Hence

$$\mu^*(A) \leq \sum_{m\in\mathbb{N}} \rho(B_m) \underset{\substack{\uparrow \\ \text{Exercise} \\ \text{1. week}}}{=} \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \rho(R_{n,k}) \right)$$

$$\leq \sum_{n=1}^{\infty} \left(\mu^*(A_n) + \varepsilon/2^n \right) = \sum_{n=1}^{\infty} \mu^*(A_n) + \underbrace{\sum_{n=1}^{\infty} \frac{\varepsilon}{2^n}}_{=\varepsilon}$$

Since $\varepsilon > 0$ is arbitrary, we get $\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$

Example/Exercise (μ^* is not always a measure):

Assume X has at least two elements, $\mathcal{R} = \{\emptyset, X\}$ and

$$\begin{cases} \rho(\emptyset) := 0 \\ \rho(X) := 1 \end{cases}$$

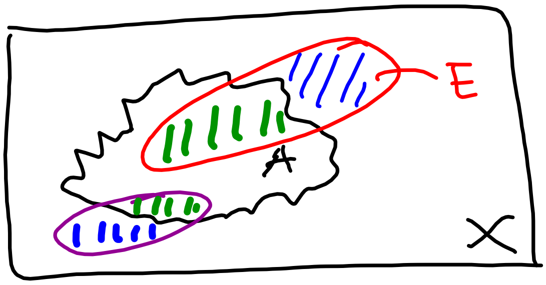
Then the associated μ^* is not a measure.

(Pick $\emptyset \neq B \neq X$ and check that $\mu^*(B) + \mu^*(B^c) = 2$, while $\mu^*(B \cup B^c) = \mu^*(X) = 1$)

Def. Let μ^* be an outer measure (on $\mathcal{P}(X)$).

We say that $A \in \mathcal{P}(X)$ is <u>μ^*-measurable</u> when we have

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad \text{for } \underline{\text{all }} E \in \mathcal{P}(X)$$



Note Since $E = (E \cap A) \cup (E \cap A^c)$, (OM3) gives that

$$\underline{\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)}$$

So to check that A is <u>μ^*-measurable</u>, it suffices to show that

$$(*) \qquad \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad \text{for all } E \in \mathcal{P}(X)$$

(and we may then assume that $\mu^*(E) < \infty$).

<u>Theorem (Carathéodory)</u>: Let μ^* be an outer measure (on $\mathcal{P}(X)$).

Set $\mathcal{A}_{\mu^*} := \{A \in \mathcal{P}(X) : A \text{ is } \mu^*\text{-measurable}\}$.

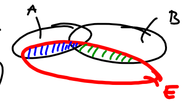
Then $\mathcal{A}_{\mu^*}$ is a σ-algebra on X. Moreover, if μ denotes the restriction of μ^* to $\mathcal{A}_{\mu^*}$, then μ is a complete measure on $\mathcal{A}_{\mu^*}$.

Proof 1) $\mathcal{A}_{\mu^*}$ is an algebra.

- $\emptyset \in \mathcal{A}_{\mu^*}$: indeed, if $E \in \mathcal{P}(X)$, then
$$\mu^*(E \cap \emptyset) + \mu^*(E \cap \emptyset^c) = \underbrace{\mu^*(\emptyset)}_{0} + \mu^*(E) = \mu^*(E)$$

- If $A \in \mathcal{A}_{\mu^*}$, then $A^c \in \mathcal{A}_{\mu^*}$ (the def. of μ^*-measurability is "symmetric" w.r.t. A and A^c)

- Assume $A, B \in \mathcal{A}_{\mu^*}$, $E \in \mathcal{P}(X)$.



Then $E \cap (A \cup B) = (E \cap A) \cup (E \cap B \cap A^c)$

Using (OM3), this implies that
$$\mu^*(E \cap (A \cup B)) \leq \mu^*(E \cap A) + \mu^*(E \cap B \cap A^c)$$
Therefore we get
$$\mu^*(E \cap (A \cup B)) + \mu^*(E \cap \underbrace{(A \cup B)^c}_{A^c \cap B^c})$$
$$\leq \mu^*(E \cap A) + \mu^*(E \cap \underbrace{B \cap A^c}_{A^c \cap B}) + \mu^*(E \cap A^c \cap B^c)$$
$$= \mu^*(E \cap A^c) \quad \text{since } B \text{ is } \mu^*\text{-meas.}$$
$$= \mu^*(E) \quad \text{since } A \text{ is } \mu^*\text{-meas.}$$

This shows that $(*)$ holds for $A \cup B$, hence that $A \cup B$ is μ^*-meas.

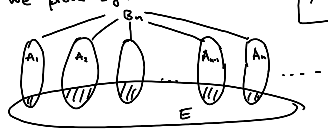
2) Next, $\mathcal{A}_{\mu^*}$ is a σ-algebra :

Since it is an algebra, it suffices to show that it is closed under countable disjoint unions.

So let $\{A_n\}$ be a seq. of disjoint sets in $\mathcal{A}_{\mu^*}$, and let $E \in \mathcal{P}(X)$.

Set $B_n := \bigcup_{j=1}^{n} A_j$, $n \in \mathbb{N}$. By 1), $B_n \in \mathcal{A}_{\mu^*}$.

We prove by induction on n that $\boxed{\mu^*(E \cap B_n) = \sum_{j=1}^{n} \mu^*(E \cap A_j)}$ $(**)$



- $n = 1$ is clear since $B_1 = A_1$.
- Assume $(**)$ holds for $n-1$, $n \geq 2$. Since A_n is μ^*-meas.

  we have
  $$\mu^*(E \cap B_n) = \mu^*(E \cap \underbrace{B_n \cap A_n}_{= A_n}) + \mu^*(E \cap \underbrace{B_n \cap A_n^c}_{= B_{n-1}})$$
  $$\overset{\text{by the ind. hyp.}}{=} \mu^*(E \cap A_n) + \sum_{j=1}^{n-1} \mu^*(E \cap A_j)$$
  $$= \sum_{j=1}^{n} \mu^*(E \cap A_j)$$

  So $(**)$ holds for n.

Now, set $A = \bigcup_{j=1}^{\infty} A_j$. Since $B_n \subseteq A$, we have $A^c \subseteq B_n^c$ $\forall n$
so $E \cap A^c \subseteq E \cap B_n^c$ $\forall n$

Hence, using (OM2), we get
$$\mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \quad (\text{since } B_n \text{ is } \mu_*\text{-meas})$$
$$\geq \mu^*(E \cap B_n) + \mu^*(E \cap A^c)$$
$$\overset{\text{using } (**)}{=} \left(\sum_{j=1}^{n} \mu^*(E \cap A_j)\right) + \mu^*(E \cap A^c) \quad \forall n$$

Letting $n \to \infty$, we get
$$\mu^*(E) \geq \left(\sum_{j=1}^{\infty} \mu^*(E \cap A_j)\right) + \mu^*(E \cap A^c)$$
$$\geq \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad (\geq \mu^*(E))$$

$E \cap A = \bigcup_{j=1}^{\infty} E \cap A_j$
$\Rightarrow \mu^*(E \cap A) \leq \sum_{j=1}^{\infty} \mu^*(E \cap A_j)$
by (OM3)

Since this holds for every $E \in \mathcal{P}(X)$, we get that A is μ^*-measurable, as desired.

3) μ is a measure on $\mathcal{A}_{\mu^*}$: Let A be as above. We see from above that
$$\mu^*(E) = \left(\sum_{j=1}^{\infty} \mu^*(E \cap A_j)\right) + \mu^*(E \cap A^c) \quad \forall E \in \mathcal{P}(X).$$

$\mu = \mu^*|_{\mathcal{A}_{\mu^*}}$

Setting $E = A$, we get $\mu^*(A) = \sum_{j=1}^{\infty} \mu^*(A_j)$, as desired.