8.3 Carathéodory's extension theorem

We recall:

Theorem Let μ^* be an outer measure (on $\mathcal{P}(X)$). Then $\mathcal{A}_{\mu^*} := \{A \subseteq X : A \text{ is } \mu^*\text{-measurable}\}$ is a σ-algebra on X and $\mu := \mu^*|_{\mathcal{A}_{\mu^*}}$ is a complete measure

Proof of completeness: Assume $A \subseteq B \subseteq X$, $B \in \mathcal{A}_{\mu^*}$ and $\mu(B) = 0$. (Have to prove that $A \in \mathcal{A}_{\mu^*}$). Note that

$$0 \leq \mu^*(A) \leq \mu^*(B) = \mu(B) = 0, \text{ so } \underline{\mu^*(A) = 0}.$$

But this implies that A is μ^*-measurable. Indeed, let $E \in \mathcal{P}(X)$. Then

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

$$\leq \underbrace{\mu^*(A)}_{=0} + \mu^*(E) = \mu^*(E),$$

since $E \cap A \subseteq A$, $E \cap A^c \subseteq E$

so $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$.

Let's go back to the situation where $\begin{cases} \rho : \mathcal{R} \to [0,\infty) \text{ and} \\ \mathcal{R} \text{ is a family of subsets of } X \end{cases}$ satisfying (i) $\emptyset \in \mathcal{R}$, (ii) $X = \bigcup_{j=1}^{\infty} R_j$ for some $\{R_j\} \subseteq \mathcal{R}$ and (iii) $\rho(\emptyset) = 0$.

We can then form an outer measure μ^* ass. to ρ, given by

$$\mu^*(A) = \inf\left\{ \sum_{n=1}^{\infty} \rho(C_n) \mid \{C_n\}_{n\in\mathbb{N}} \text{ is an } \mathcal{R}\text{-covering of } A \right\}$$

for $A \subseteq X$.

We can apply Carathéodory's theorem to μ^*. But the trouble is that it may happen that $\mathcal{R}$ is not contained in $\mathcal{A}_{\mu^*}$. We need some assumptions to garantee this.

Def. We say that ρ is a premeasure on $\mathcal{R}$ when $\rho(\emptyset) = 0$ and $\rho\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \rho(A_j)$ whenever $\{A_j\}$ is a sequence of disjoint sets in $\mathcal{R}$ such that $\bigcup_{j=1}^{\infty} A_j \in \mathcal{R}$.

Note that, as for measures, we get that

$$\rho(A \cup B) = \rho(A) + \rho(B) \quad \text{whenever } A, B \in \mathcal{R},\ A \cap B = \emptyset \text{ and } A \cup B \in \mathcal{R}$$

and $\rho(A) \le \rho(B)$ whenever $A, B \in \mathcal{R}$ and $A \subseteq B$.

Carathéodory's extension theorem (for algebras)

Assume $\mathcal{R}$ is an algebra (on X) and $\rho : \mathcal{R} \to [0, \infty]$ is a premeasure on $\mathcal{R}$. Let μ^* denote the outer measure ass. with $\mathcal{R}$ and ρ.

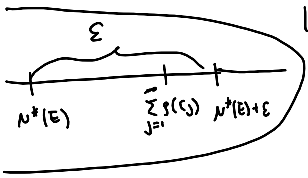
Then $\mathcal{R} \subseteq \Sigma(\mathcal{R}) \subseteq \mathcal{A}_{\mu^*}$.

Moreover, $\mu := \mu^*|_{\mathcal{A}_{\mu^*}}$ (and $\mu^*|_{\Sigma(\mathcal{R})}$) agrees with ρ on $\mathcal{R}$ (so μ is an "extension" of ρ to $\mathcal{A}_{\mu^*}$ (and to $\Sigma(\mathcal{R})$)).

Proof • We first show that $\mathcal{R} \subseteq \mathcal{A}_{\mu^*}$ (this will imply that $\Sigma(\mathcal{R}) \subseteq \mathcal{A}_{\mu^*}$).

Let $A \in \mathcal{R}$. To show that $A \in \mathcal{A}_{\mu^*}$, we consider $E \in \mathcal{P}(X)$ such that $\mu^*(E) < \infty$. We must show that

$$\mu^*(E \cap A) + \mu^*(E \cap A^c) \le \mu^*(E) \quad (*).$$



Let $\varepsilon > 0$. Then we can pick an $\mathcal{R}$-covering of E, say $\{C_j\}_{j=1}^{\infty}$, such that

$$\sum_{j=1}^{\infty} \rho(C_j) < \mu^*(E) + \varepsilon$$

Since ρ is a premeasure on $\mathcal{R}$, we have that

$$\rho(C_j) = \rho(\underbrace{C_j \cap A}_{\in \mathcal{R}}) + \rho(\underbrace{C_j \cap A^c}_{\in \mathcal{R}}) \quad \forall j$$

Moreover, since

$$\begin{cases} E \cap A \subseteq \bigcup_{j=1}^{\infty} C_j \cap A & (\text{since } E \subseteq \bigcup_{j=1}^{\infty} C_j) \\ E \cap A^c \subseteq \bigcup_{j=1}^{\infty} C_j \cap A^c & (\text{— '' —}) \end{cases}$$

we also have

$$\begin{cases} \mu^*(E \cap A) \le \sum_{j=1}^{\infty} \rho(C_j \cap A) \\ \mu^*(E \cap A^c) \le \sum_{j=1}^{\infty} \rho(C_j \cap A^c) \end{cases}$$

Hence we get

$$\mu^*(E \cap A) + \mu^*(E \cap A^c) \le \sum_{j=1}^{\infty} \left(\rho(C_j \cap A) + \rho(C_j \cap A^c) \right) = \sum_{j=1}^{\infty} \rho(C_j)$$

$$\le \mu^*(E) + \varepsilon.$$

This holds for every $\varepsilon > 0$, so we get that $(*)$ holds, as desired

• It remains to show that $\mu^*(A) = \rho(A)$ when $A \in \mathcal{R}$.

So let $A \in \mathcal{R}$. We have that

$\mu^*(A) \leq \rho(A)$ (since $A, \emptyset, \emptyset, \ldots$ is an $\mathcal{R}$-covering of A)

To show the reverse inequality, let $\{C_j\}_{j=1}^{\infty}$ be an $\mathcal{R}$-covering of A. Define
$$\begin{cases} D_1 := A \cap C_1 \\ D_k := A \cap \left(C_k \setminus \bigcup_{j=1}^{k-1} C_k\right), \quad k \geq 2 \end{cases}$$

Then $\{D_k\}_{k \in \mathbb{N}}$ is a seq. of disjoint sets in $\mathcal{R}$ such that $D_k \subseteq C_k$ for each k and
$$\bigcup_{k=1}^{\infty} D_k = A \cap \underbrace{\left(\bigcup_{j=1}^{\infty} C_j\right)}_{\supseteq A} = A$$

Since ρ is a premeasure, we get
$$\rho(A) = \sum_{k=1}^{\infty} \rho(D_k) \leq \sum_{k=1}^{\infty} \rho(C_k).$$

Since this holds for all $\mathcal{R}$-coverings $\{C_j\}_{j=1}^{\infty}$ of A, we get that

$\rho(A) \leq \mu^*(A)$, as desired.

In applications, it is more useful to have at hand a premeasure on a semialgebra:

Def. Let $\mathcal{S} \subseteq \mathcal{P}(X)$, $\mathcal{S} \neq \emptyset$. We say that $\mathcal{S}$ is a semialgebra (on X) if the foll. conditions hold:

(i) $S, S' \in \mathcal{S} \Rightarrow S \cap S' \in \mathcal{S}$.

(ii) If $S \in \mathcal{S}$, then S^c is a disjoint union of finitely many sets in $\mathcal{S}$.

Note: We have $\emptyset \in \mathcal{S}$: Let $S \in \mathcal{S}$. Then

$S^c = S_1 \cup \cdots \cup S_n$ for some disjoint $S_1, \ldots, S_n$ in $\mathcal{S}$.

So $\emptyset = S \cap S_1 \in \mathcal{S}$

This gives that $X = \emptyset^c$ is a disjoint union of finitely many sets in $\mathcal{S}$.

Exercise: Let $\mathcal{J}$ be the family of half-open intervals in $\mathbb{R}$ (open to the left) as in a prev. Extra Exercise. Then $\mathcal{J}$ is a semialgebra on $\mathbb{R}$.

Carathéodory's extension theorem (for semi algebras)

> Assume $\lambda : \mathcal{S} \to [0,\infty)$ is a premeasure on a semialg. $\mathcal{S}$ (on X).
> Then λ has an extension to a complete measure μ on a $\underbrace{\sigma\text{-algebra}}_{\mathcal{A}}$ which contains $\mathcal{S}$.

Proof: The proof goes as follows.

Step 1: Let $\mathcal{R}$ consist of all finite disjoint union of sets in $\mathcal{S}$. Then $\mathcal{R}$ is an algebra (on X) containing $\mathcal{S}$.

Step 2 For $A \in \mathcal{R}$, say $A = S_1 \cup \dots \cup S_n$ (where each $S_j \in \mathcal{S}$ and $S_j \cap S_k = \emptyset$ for $j \neq k$.)

we set $\rho(A) := \sum_{j=1}^{n} \lambda(S_j)$

Then $\rho : \mathcal{R} \to [0,\infty]$ is $\underbrace{\text{a}}_{\text{well-defined}}$ premeasure on $\mathcal{R}$

which extends λ (and is the unique premeasure on $\mathcal{R}$ which extends λ)

Step 3 We apply Carathéodory's theorem (for algebras) to $\mathcal{R}$ and ρ.

Letting μ^* be the outer measure (on $\mathcal{P}(X)$) ass. to $\mathcal{R}$ and ρ

we get that $\mathcal{S} \subseteq \mathcal{R} \subseteq \mathcal{A}_{\mu^*}$ and

$\mu = \mu^*|_{\mathcal{A}_{\mu^*}}$ is a complete measure on $\mathcal{A}_{\mu^*}$

extending ρ, hence extending λ.