(8.3) More on Carathéodory's extension theorem

Assume $\mathcal{S}$ is a semialgebra on $X\,(\neq \emptyset)$ and $\lambda : \mathcal{S} \to [0,\infty]$ is a premeasure.

$S, S' \in \mathcal{S} \Rightarrow S \cap S' \in \mathcal{S}$
$S \in \mathcal{S} \Rightarrow S^c = S_1 \cup \dots \cup S_n$ for some $S_1, \dots, S_n$ in $\mathcal{S}$, $S_j \cap S_k = \emptyset$ for $j \neq k$
Note: $\emptyset \in \mathcal{S}$

$\lambda(\emptyset) = 0$ and
$\lambda(\bigcup_{j=1}^\infty S_j) = \sum_{j=1}^\infty \lambda(S_j)$
when $\{S_j\} \subseteq \mathcal{S}$ is a disjoint family and $\bigcup_{j=1}^\infty S_j \in \mathcal{S}$

We have seen that λ has an extension to a complete measure μ on a σ-alg. $\mathcal{A}$ containing $\mathcal{S}$ (hence containing $\sigma(\mathcal{S})$)

$\sigma(\mathcal{S})$: the σ-alg. generated by $\mathcal{S}$.

We will show that, under a certain condition, μ is unique.

We recall that $\mathcal{R} := \{$all finite disjoint union of sets in $\mathcal{S}\}$ is an algebra containing $\mathcal{S}$ and $\rho : \mathcal{R} \to [0,\infty]$ def. by

$$\rho\Big(\bigcup_{j=1}^n S_j\Big) := \sum_{j=1}^n \lambda(S_j) \quad \text{when } S = S_1 \cup \dots \cup S_n,\ S_1, \dots, S_n \in \mathcal{S} \text{ and } S_j \cap S_k = \emptyset,\ j \neq k.$$

is a premeasure on $\mathcal{R}$.

Letting μ^* be the outer measure ass. to $\mathcal{R}$ and ρ, and setting $\mathcal{A} := \mathcal{A}_{\mu^*}$, the measure μ on $\mathcal{A}$ is given by $\mu := \mu^*|_{\mathcal{A}}$.

$\mathcal{A} \supseteq \mathcal{R} \supseteq \mathcal{S}$

Note in passing that $\mu^* = \nu^*$ on $\mathcal{P}(X)$, where ν^* is the outer measure ass. to $\mathcal{S}$ and λ. (Exercise!)

Proposition: Assume that μ is σ-finite, that is, there exists $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ such that $X = \bigcup_{n=1}^\infty A_n$ and $\mu(A_n) < \infty$ $\forall n \in \mathbb{N}$.

Then μ is the only measure on $\mathcal{A}$ extending λ.

(Similarly, $\mu|_{\sigma(\mathcal{S})}$ is the only measure on $\sigma(\mathcal{S})$ extending λ).

Proof. We will first show the following general fact:

(*) $\boxed{\begin{array}{l}\text{Assume } \mathcal{B} \text{ is a } \sigma\text{-algebra on } X \text{ containing } \mathcal{R} \\ \text{and } \nu \text{ is a measure on } \mathcal{B} \text{ extending } \rho. \\ \text{Let } A \in \mathcal{A} \cap \mathcal{B}. \text{ Then we have} \\ \nu(A) \le \mu(A), \text{ and } \nu(A) = \mu(A) \text{ if } \mu(A) < \infty \end{array}}$

Proof of (*): To show this, let $\{C_j\}_{j\in\mathbb{N}}$ be an $\mathcal{R}$-cover of A. Then

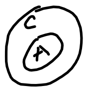
$$\underset{\substack{\uparrow \\ \nu \text{ monotone}}}{\nu(A) \le} \nu\Big(\bigcup_{j=1}^{\infty} C_j\Big) \underset{\substack{\uparrow \\ \nu \text{ subadd.}}}{\le} \sum_{j=1}^{\infty} \nu(C_j) = \sum_{j=1}^{\infty} \rho(C_j)$$

Taking the inf. over all $\mathcal{R}$-covers of A, we get $\underline{\nu(A) \le \mu^*(A) = \mu(A)}$.

Assume now that $\underline{\mu(A) < \infty}$, i.e. $\mu^*(A) < \infty$. We can pick an $\mathcal{R}$-cover $\{C_j\}_{j\in\mathbb{N}}$ of A s.t. $\sum_{j=1}^{\infty} \rho(C_j) < \infty$, and we can assume that the C_j's are disjoint $\left(\text{if not, use } \begin{cases} C_1' = C_1 \\ C_k' = C_k \setminus \bigcup_{j=1}^{k} C_j, & k \ge 2 \end{cases}\right)$

Set $C := \bigcup_{j=1}^{\infty} C_j \in \mathcal{A} \cap \mathcal{B}$. Then

$$\nu(C) \underset{\substack{\uparrow \\ \nu \text{ is a measure}}}{=} \sum_{j=1}^{\infty} \nu(C_j) = \sum_{j=1}^{\infty} \rho(C_j) = \sum_{j=1}^{\infty} \mu(C_j) \underset{\substack{\uparrow \\ \mu \text{ is a measure}}}{=} \mu(C).$$



Thus $\quad \underline{\nu(A) + \nu(C\setminus A)} = \nu(C) = \mu(C) = \underline{\mu(A) + \mu(C\setminus A)}$

But $\nu(A) \le \mu(A)$ and $\nu(C\setminus A) \le \mu(C \setminus A)$ $\left(\text{since } \underbrace{C\setminus A}_{= C\cap A^c} \in \mathcal{A}\cap\mathcal{B}\right)$

This implies that $\underline{\nu(A) = \mu(A)}$, as desired.

We can now prove the proposition.

Assume $\nu : \mathcal{A} \to [0,\infty]$ is a measure extending λ.

Note that $\underline{\nu \text{ extends } \rho}$: indeed, if $R = S_1 \cup \dots \cup S_n$ where $S_1,\dots,S_n \in \mathcal{S}$, $S_j \cap S_k \ne \emptyset$ for $j \ne k$

then $$\nu(R) \underset{\substack{\uparrow \\ \nu \text{ is a measure}}}{=} \sum_{j=1}^{n} \nu(S_j) = \sum_{j=1}^{n} \lambda(S_j) \underset{\substack{\uparrow \\ \text{by def. of } \rho}}{=} \rho(R).$$

By assumption, $X = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathcal{A}$, $\mu(A_n) < \infty$ for all n.

Set $B_n := \bigcup_{j=1}^{n} A_j \in \mathcal{A}$ for each $n \in \mathbb{N}$.

Then $B_n \subseteq B_{n+1}$, $\mu(B_n) \le \sum_{j=1}^{n} \mu(A_j) < \infty$ for all n

and $\bigcup_{n=1}^{\infty} B_n = X$.

$\underline{\text{Let } A \in \mathcal{A}}$. Then $A = A \cap X = A \cap \Big(\bigcup_{n=1}^{\infty} B_n\Big) = \bigcup_{n=1}^{\infty} (A \cap B_n)$

and $\underline{A\cap B_n \subseteq A \cap B_{n+1}}$, $\underline{\mu(A\cap B_n) \le \mu(B_n) < \infty}$.

Then the fact (*) proven above (with $\mathcal{B} = \mathcal{A}$) gives that

$$\nu(A\cap B_n) = \mu(A \cap B_n) \quad \text{for all } n.$$

Using $\underline{\text{cont. from below of } \nu \text{ and } \mu}$, we get

$$\underline{\nu(A)} = \nu\Big(\bigcup_{n=1}^{\infty} (A\cap B_n)\Big) = \lim_{n\to\infty} \underbrace{\nu(A\cap B_n)}_{= \mu(A\cap B_n)} = \mu\Big(\bigcup_{n=1}^{\infty} (A\cap B_n)\Big) = \underline{\mu(A)}$$

Thus $\nu = \mu$, as desired.

8.4 Lebesgue measure on $\mathbb{R}$

To construct Lebesgue measure we will apply Car. ext. theorem to $\mathcal{S}$ and λ, where:

$$\mathcal{S} = \{\emptyset\} \cup \{(a,b] : a<b,\, a,b\in\mathbb{R}\} \cup \{(a,\infty) : a\in\mathbb{R}\} \cup \{(-\infty,b] : b\in\mathbb{R}\}$$

and $\lambda : \mathcal{S} \to [0,\infty]$ is def. by

$$\begin{cases} \lambda(\emptyset) := 0 \\ \lambda((a,b]) := b-a \\ \lambda((a,\infty)) = \lambda((-\infty,b]) := \infty \end{cases}$$

We leave the proof that $\mathcal{S}$ is semialgebra on $\mathbb{R}$ as an exercise.

(f.ex. $a \quad a' \quad b \quad b'$ $\quad = (a,b] \cap (a',b']$ $\qquad$ $(a,b]^c = \underbrace{(-\infty,a]}_{\in\mathcal{S}} \cup \underbrace{(b,\infty)}_{\in\mathcal{S}}$)

The nontrivial part is to show that

Prop. $\boxed{\lambda \text{ is a premeasure on } \mathcal{S}}$

Proof Let $\{S_j\}_{j\in\mathbb{N}}$ be a disjoint seq. in $\mathcal{S}$ such that $S := \bigcup_{j=1}^{\infty} S_j \in \mathcal{S}$. We must show that $\boxed{\lambda(S) = \sum_{j=1}^{\infty}\lambda(S_j)}$ $(*)$

We first note that λ is finitely additive on $\mathcal{S}$, that is,

$\lambda(T) = \sum_{j=1}^{n}\lambda(T_j)$ whenever $T_1,\dots,T_n \in \mathcal{S}$, $T_j\cap T_k = \emptyset$ for $j\neq k$ and $T := \bigcup_{j=1}^{n} T_j \in \mathcal{S}$.

Note that if T is unbounded, then at least one of the T_j's must be unbounded, so both $\lambda(T)$ and $\sum_{j=1}^{n}\lambda(T_j)$ are infinite.
If T is bounded, we have $\quad T_j \quad a \quad b \quad \underbrace{\qquad\qquad}_{T}$
and it is clear that $\lambda(T) = \sum_{j=1}^{n}\lambda(T_j)$

Set $\mathcal{R} := \{\text{all finite union of disjoint sets in } \mathcal{S}\}$ and define $\rho : \mathcal{R} \to [0,\infty)$ as in sect. 8.3. As in 8.3, we have that ρ is well-defined, is finitely sub-additive on $\mathcal{R}$, and is monotone.

To prove $(*)$, consider first the case where $S = \emptyset$.
Then all S_j's are empty, so $(*)$ is trivial.
So assume $S \neq \emptyset$.

Then we have $\lambda(S) = \rho(S) \geq \rho\left(\bigcup_{j=1}^{n} S_j\right) = \sum_{j=1}^{n}\rho(S_j) = \sum_{j=1}^{n}\lambda(S_j)$

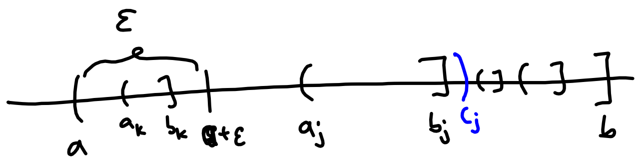
(since ρ is monotone; ρ fin. add.)

Letting $n\to\infty$, we get $\lambda(S) \geq \sum_{j=1}^{\infty}\lambda(S_j)$

To show the reverse inequality $\sum_{j=1}^{\infty} \lambda(S_j) \leq \lambda(S)$,

Consider $S = (a, b]$, $a < b$.

Then $S_j = (a_j, b_j]$ for some $a \leq a_j < b_j \leq b$ for each $j \in \mathbb{N}$.



Let $\varepsilon > 0$, $\varepsilon < b - a$. For each $j \in \mathbb{N}$, set $c_j := b_j + \frac{\varepsilon}{2^j}$.

Then we have

$$[a+\varepsilon, b] \subseteq \underbrace{(a, b]}_{S} = \bigcup_{j=1}^{\infty} \underbrace{(a_j, b_j]}_{S_j} \subseteq \bigcup_{j=1}^{\infty} (a_j, c_j)$$

So $\{(a_j, c_j)\}_{j \in \mathbb{N}}$ is an open cover of $[a+\varepsilon, b]$ which is compact.

So we can pick a finite subcover. Thus we can find $N \in \mathbb{N}$

such that $[a+\varepsilon, b] \subseteq \bigcup_{j=1}^{N} (a_j, c_j)$

This implies that $(a+\varepsilon, b] \subseteq \bigcup_{j=1}^{N} (a_j, c_j]$

Hence we get

$$\underbrace{b-a}_{\lambda(S)} - \varepsilon = \lambda((a+\varepsilon, b]) = \rho((a+\varepsilon, b]) \overset{\rho \text{ monotone}}{\leq} \rho\Big(\bigcup_{j=1}^{N} (a_j, c_j]\Big)$$

$$\overset{\rho \text{ is finitely subadd.}}{\leq} \sum_{j=1}^{N} \rho((a_j, c_j]) = \sum_{j=1}^{N} (c_j - a_j) = \sum_{j=1}^{N} \Big(b_j + \frac{\varepsilon}{2^j} - a_j\Big)$$

$$= \underbrace{\sum_{j=1}^{N} (b_j - a_j)}_{\sum_{j=1}^{N} \lambda(S_j)} + \underbrace{\sum_{j=1}^{N} \varepsilon/2^j}_{< \varepsilon} < \sum_{j=1}^{\infty} \lambda(S_j) + \varepsilon$$

Thus $\lambda(S) < \sum_{j=1}^{\infty} \lambda(S_j) + 2\varepsilon$.

This holds for every $\varepsilon > 0$, so we get $\lambda(S) \leq \sum_{j=1}^{\infty} \lambda(S_j)$, as desired.

$S = (a, \infty)$ or $(-\infty, b]$: next time!