We finished the proof that λ is a premeasure on $\mathcal{S}$ (see the updated notes for Tuesday).

So we can apply Carathéodory's ext. theorem to $\mathcal{S}$ and λ:

We get that λ can be extended to a complete measure μ on a σ-algebra $\mathcal{L}$ containing $\mathcal{S}$ (hence $\sigma(\mathcal{S}) = \mathcal{B}_{\mathbb{R}}$) which consists of the subsets of $\mathbb{R}$ which are λ^*-measurable.

λ^*: the outer measure on $\mathcal{P}(\mathbb{R})$ ass. to $\mathcal{S}$ and λ

The measure μ is called the Lebesgue measure on $(\mathbb{R}, \mathcal{L})$ (or on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$) while sets in $\mathcal{L}$ are called Lebesgue measurable.

Note that μ is σ-finite: indeed,

$$\mathbb{R} = \bigcup_{n=1}^{\infty} (-n, n] \quad \text{and} \quad \mu((-n,n]) = \lambda((-n,n]) = 2n < \infty$$

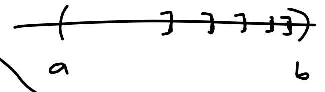
for every $n \in \mathbb{N}$.

This implies that μ is the only extension of λ to a measure on $\mathcal{L}$ (and on $\mathcal{B}_{\mathbb{R}}$).

Note that $\mu(I) =$ length of I for any interval $I \subseteq \mathbb{R}$ and $\mu(\{b\}) = 0 \quad \forall b \in \mathbb{R}$:

F.ex. Let $a, b \in \mathbb{R}$, $a<b$.

Since $(a,b) = \bigcup_{n=1}^{\infty} \left(a, b - \frac{(b-a)}{n}\right]$



we get that

$$\mu((a,b)) = \lim_{n\to\infty} \underbrace{\mu\left(\left(a, b - \tfrac{(b-a)}{n}\right]\right)}_{\lambda\left(\left(a, b - \frac{(b-a)}{n}\right]\right)}$$

$$= \lim_{n\to\infty} b-a - \frac{(b-a)}{n} = b-a$$

Thus $\mu(\{b\}) = \mu((a,b] \setminus (a,b))$

$= \mu((a,b]) - \mu((a,b)) = b-a-(b-a) = 0$

etc...

The Lebesgue measure has the foll. useful properties:

Prop. Let $A \subseteq \mathbb{R}$ be Lebesgue measurable, $x \in \mathbb{R}$, $r \in \mathbb{R}$.

Then $A+x := \{a+x : a \in A\}$ is leb. measurable

and $\underline{\mu(A+x) = \mu(A)}$ ("μ is translation-invariant")

Moreover, $rA := \{ra : a \in A\}$ is leb. measurable

and $\underline{\mu(rA) = |r|\,\mu(A)}$

(We can replace leb. measurability by Borel measurability)

Proof Exercise!

Note There are subsets of $\mathbb{R}$ which are not Lebesgue-measurable!

An example is as follows.

Consider $X = [0,1)$. Define an equivalence relation on X

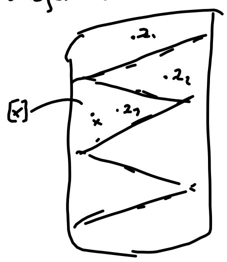
by $\boxed{x \sim y \iff x-y \in \mathbb{Q}}$

For $x \in X$, set $[x] := \{y \in X : x \sim y\}$. Recall that

↑ the equivalence class of x

any two equivalence classes are either equal or disjoint.

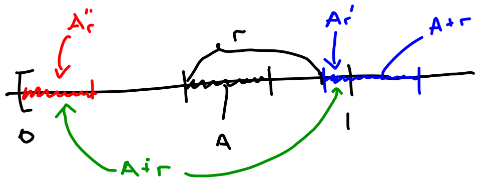
Let's pick one element in each of the equivalence classes and let Z consist of all these elements.



(Then we have $X = \bigcup_{z \in Z} [z]$ (disj. union))

Will argue that Z is not Lebesgue-measurable.

If $A \subseteq X$ and $r \in X$, set $A \dot{+} r = \{a+r \ (\text{mod } 1),\ a \in A\}$



$= \overbrace{\{a+r : a \in A \cap [0,1-r)\}}^{A_r'}$

$\cup \underbrace{\{a+r-1 : a \in A \cap [1-r,1)\}}_{A_r''}$

Set $R = \mathbb{Q} \cap [0,1)$.

Claim $\boxed{\text{Each } x \in X \text{ belongs to } Z \dot{+} r \text{ for exactly one } r \in R}$.

(check this if you are in the right mood).

This will give that $\underbrace{[0,1)}_{X} = \bigcup_{r \in R} Z \dot{+} r$ (disjoint union)

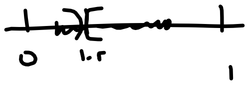
So if we assume (for contradiction) that Z is Leb. measurable,

we get that $Z \dot{+} r = Z_r' \cup Z_r''$ is Leb. measurable $\forall r \in R$

are Leb. meas. (by using the prev. prop.)

and
$$\mu(Z \dot{+} r) = \mu(Z_r') + \mu(Z_r'')$$
$$= \mu\big((Z \cap [0,1-r)) + r \big) + \mu\big((Z \cap [1-r,1)) + r - 1 \big)$$
$$= \mu(Z \cap [0,1-r)) + \mu(Z \cap [1-r,1))$$

μ is transl. invariant $= \mu(Z) \quad \forall r \in R$



Thus we get that
$$1 = \mu([0,1)) = \mu\Big(\bigcup_{r \in R} Z \dot{+} r \Big) = \sum_{r \in R} \underbrace{\mu(Z \dot{+} r)}_{\mu(Z)}$$
$$= \begin{cases} 0 & \text{if } \mu(Z) = 0 \\ \infty & \text{if } \mu(Z) > 0 \end{cases}$$

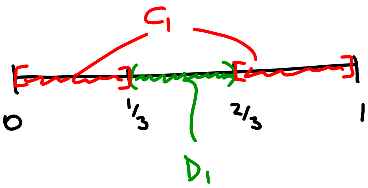
a contradiction! So Z can not be Leb. measurable.

(The proof shows that there is no measure on $\mathcal{P}(\mathbb{R})$ such that $\mu([0,1)) = 1$ which is transl. invariant.)

The Cantor set C

We start with $[0,1]$

Step 1



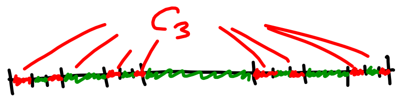
$D_1 = (1/3, 2/3)$ $C_1 = [0,1] \setminus D_1$

Step 2

$D_2 = (1/3, 2/3) \cup (1/9, 2/9) \cup (7/9, 8/9)$ $C_2 = [0,1] \setminus D_2$

$= D_1 \cup \frac{1}{3} D_1 \cup \left(\frac{2}{3} + \frac{1}{3} D_1\right)$

Step 3



$D_3 = D_1 \cup \frac{1}{3} D_2 \cup \left(\frac{2}{3} + \frac{1}{3} D_2\right)$

$C_3 = [0,1] \setminus D_3$

⋮

Step k $D_k := D_1 \cup \frac{1}{3} D_{k-1} \cup \left(\frac{2}{3} + \frac{1}{3} D_{k-1}\right)$ $C_k = [0,1] \setminus D_k$

Set $D := \bigcup_{k=1}^{\infty} D_k$, $\underline{C := [0,1] \setminus D}$ Cantor set.

D is open, hence Leb. meas. So $\underline{C \text{ is Leb. meas}}$ and closed (therefore compact)

Note that $\mu(D_k) = \mu(D_1) + \mu\left(\frac{1}{3} D_{k-1}\right) + \mu\left(\frac{2}{3} + \frac{1}{3} D_{k-1}\right)$

$$= \mu(D_1) + \frac{2}{3} \mu(D_{k-1})$$

Since $D_1 \subseteq D_2 \subseteq \cdots \subseteq D_k \subseteq D_{k+1} \subseteq \cdots \subseteq D = \bigcup_k D_k$,

we have $\mu(D) = \lim_{k \to \infty} \mu(D_k) = \lim_{k \to \infty} \mu(D_1) + \frac{2}{3} \mu(D_{k-1})$

$$= 1/3 + \frac{2}{3} \mu(D)$$

$$\Rightarrow \underline{\mu(D) = 1} \Rightarrow \mu(C) = \mu([0,1]) - \mu(D)$$

$$= 1 - 1 = \underline{\underline{0}}$$

So C is "thin".

But there is a bijection from C onto $[0,1]$! So it is also "big"!

↑

see in Brevig's note.