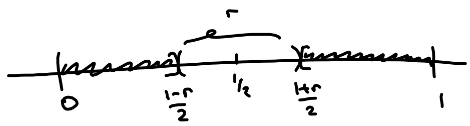
Other "Cantor like" sets

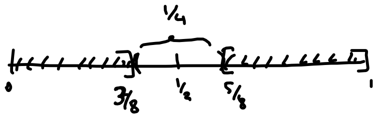
Consider $0 < r < 1/3$. As with the Cantor set (with $r = \frac{1}{3}$) we can first remove from $[0,1]$ the open interval of length r around the middle. We are left with



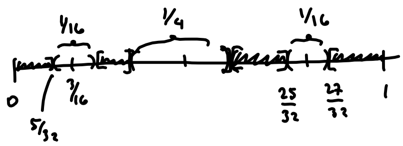
Then we can repeat this. At each step we remove the open middle interval of length r^n. At the end we get the "fat" Cantor set F_r

F.ex. with $\boxed{r = \frac{1}{4}}$

Step 1



Step 2



- ...

At step n, we will have removed open intervals of total length

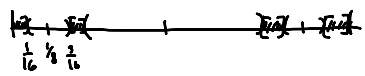
$$\frac{1}{4} + 2 \cdot \frac{1}{16} + \cdots + 2^{n-1} \frac{1}{4^n}$$

So the Leb. measure of $F_r = 1 - \sum_{n=1}^{\infty} 2^{n-1} \frac{1}{4^n} = 1 - \frac{1}{2} \underbrace{\sum_{n=1}^{\infty} \frac{1}{2^n}}_{1} = \underline{\underline{\frac{1}{2}}}$

Another variant is to remove a fixed percentage from the middle at each step. Let's say $1/2$ (instead of $1/3$ for the Cantor set).

Step 1

Step 2



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The total length of all removed intervals will be

$$\frac{1}{2} + 2 \cdot \frac{1}{8} + \cdots + 2^{n-1} \frac{1}{2 \cdot 4^n} + \cdots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$$

So the Cantor-like set we get has Leb. measure $1 - 1 = \underline{\underline{0}}$

Lebesgue's theorem about Riemann-integrability:

Assume $f:[a,b]\to\mathbb{R}$ is bounded.

Then $D_f := \{x\in[a,b] : f \text{ is not cont. at } x\}$

Then $\boxed{f \text{ is Riemann-integrable} \iff \lambda^*(D_f)=0 \quad (\iff D_f \text{ is leb. meas. and } \lambda(D_f)=0)}$

λ^*: the outer measure on $\mathcal{P}(\mathbb{R})$ associated with length of half-open intervals

Note: If $A\subseteq\mathbb{R}$ then $\lambda^*(A)=0$ $\Updownarrow$ A is leb. meas. and $\lambda(A)=0$

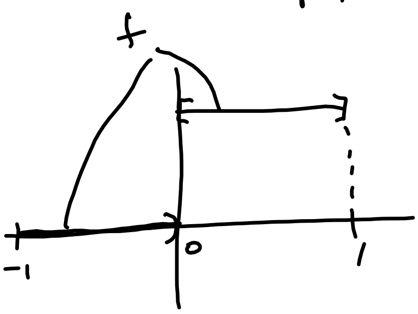
This characterization is sometimes useful.

See f. ex. the "pop corn"-function in Breivig's notes (sect. 1.2)

Another example: Assume $f:[a,b]\to\mathbb{R}$ is bounded and monotone (increasing or decreasing). Then D_f is countable (exercise). So D_f is leb. meas. and has measure zero. Hence f is Riemann-integrable (by the theorem above).

Note: The theorem does not say that Riem.-integrability of f is equivalent to $f=g$ a.e. for some continuous g on $[a,b]$.

For example, $f=1_{[0,1]}$ is Riemann-integrable on $[-1,1]$.



But there exists no continuous $g:[-1,1]\to\mathbb{R}$ s.t. $f=g$ a.e. (w.r.t. leb. measure)

(Exercise!)

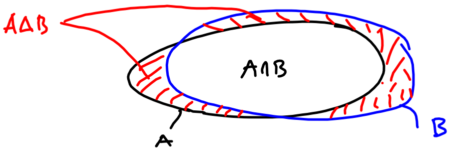
2.1 (in Brevig's notes)

Littlewood's first principle – approximation of leb. measurable sets.

$\mathcal{L}$ = leb. measurable subsets of $\mathbb{R}$, μ = leb. measure on $(\mathbb{R}, \mathcal{L})$

Notation: If $A, B \subseteq X$, then $\boxed{A \Delta B := (A \setminus B) \cup (B \setminus A)}$

the "symmetric difference"

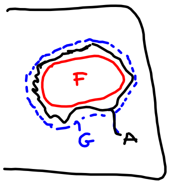


Theorem (Littlewood's first principle) Let $A \in \mathcal{L}$ and $\varepsilon > 0$.

1) Assume $\mu(A) < \infty$. Then there exists a finite union of bounded intervals B such that $\mu(A \Delta B) < \varepsilon$.

2) There exist $F \subseteq \mathbb{R}$ closed and $G \subseteq \mathbb{R}$ open such that

$F \subseteq A \subseteq G$ and $\begin{cases} \mu(A \setminus F) < \varepsilon \\ \mu(G \setminus A) < \varepsilon \end{cases}$



Note: In 1), the intervals may be of any type.

Proof Assume first that $\mu(A) < \infty$.

Since $\mu(A) = \lambda^*(A) < \infty$, we can find a sequence $\{(a_j, b_j]\}_{j \in \mathbb{N}}$ (where $-\infty < a_j \le b_j < \infty$) [with the convention that $(a,a] = \emptyset \ \forall a \in \mathbb{R}$]

such that $A \subseteq \bigcup_{j \in \mathbb{N}} (a_j, b_j]$ and

$$\sum_{j=1}^{\infty} (b_j - a_j) < \mu(A) + \varepsilon/2 . \quad (*)$$

We can now pick $J \in \mathbb{N}$ s.t. $\sum_{j=J+1}^{\infty} (b_j - a_j) < \varepsilon/2$,

and set $B := \bigcup_{j=1}^{J} (a_j, b_j]$.

Then we have $A \setminus B \subseteq \left(\bigcup_{j \in \mathbb{N}} (a_j, b_j]\right) \setminus \left(\bigcup_{j=1}^{J} (a_j, b_j]\right)$

$$\subseteq \bigcup_{j=J+1}^{\infty} (a_j, b_j]$$

So $\mu(A \setminus B) \le \sum_{j=J+1}^{\infty} (b_j - a_j) < \varepsilon/2$.

Moreover, $B \setminus A = \left(\bigcup_{j=1}^{J} (a_j, b_j]\right) \setminus A \subseteq \left(\bigcup_{j=1}^{\infty} (a_j, b_j]\right) \setminus A$

So $\mu(B \setminus A) \le \mu\left(\left(\bigcup_{j=1}^{\infty} (a_j, b_j]\right) \setminus A\right)$

$$= \mu\left(\bigcup_{j=1}^{\infty} (a_j, b_j]\right) - \mu(A)$$

since $\mu(A) < \infty$

$$\le \sum_{j=1}^{\infty} (b_j - a_j) - \mu(A) < \cancel{\mu(A)} + \varepsilon/2 - \cancel{\mu(A)}$$

Hence $\mu(A \Delta B) = \mu(A \setminus B) + \mu(B \setminus A) < \varepsilon/2 + \varepsilon/2 = \varepsilon$,

showing that 1) holds.

Set now $G := \bigcup_{j=1}^{\infty} (a_j, b_j + \varepsilon/2^{j+1})$, which is open in $\mathbb{R}$ and contains A.

Then $\mu(G) \leq \sum_{j=1}^{\infty} \left(b_j - a_j + \frac{\varepsilon}{2^{j+1}}\right) = \left(\sum_{j=1}^{\infty} (b_j - a_j)\right) + \frac{1}{2}\cdot\varepsilon$

$< \mu(A) + \underbrace{\varepsilon/2 + \varepsilon/2}_{=\varepsilon}$

Hence $\mu(G \setminus A) = \mu(G) - \mu(A) < \varepsilon$.

Let's now consider the case where $\mu(A) = \infty$.

Set $A_n := A \cap (n, n+1]$, $n \in \mathbb{Z}$

Then $\mu(A_n) \leq 1$ for all $n \in \mathbb{Z}$. So by using what we have just shown, we can find $G_n \subseteq \mathbb{R}$ open such that

$A_n \subseteq G_n$ and $\mu(G_n \setminus A_n) < \varepsilon/2^{|n|+2}$ for all $n \in \mathbb{Z}$

Set $G := \bigcup_{n \in \mathbb{Z}} G_n$ open in $\mathbb{R}$. Then

$A = \bigcup_{n \in \mathbb{Z}} A_n \subseteq G$ and $G \setminus A \subseteq \bigcup_{n \in \mathbb{Z}} (G_n \setminus A_n)$

if $x \in G \setminus A$, then $x \in G_n$ for some n and $x \notin A_k$ for all k.

So $\mu(G \setminus A) \leq \sum_{n \in \mathbb{Z}} \mu(G_n \setminus A_n)$

$\leq \sum_{n \in \mathbb{Z}} \varepsilon/2^{|n|+2} = \frac{\varepsilon}{4}(1 + 1 + 1) < \varepsilon$.

Thus we have shown 2) for G.

Finally, we show 2) for F:

We consider $A^c \in \mathcal{L}$. Using what we have done above, we can find $H \subseteq \mathbb{R}$ open s.t. $A^c \subseteq H$ and $\mu(H \setminus A^c) < \varepsilon$.

Set $F := H^c$, which is closed, $\underbrace{H^c}_{F} \subseteq \underbrace{(A^c)^c}_{A}$ and

$A \setminus F = A \cap \underbrace{F^c}_{H} = H \cap (A^c)^c = H \setminus A^c$,

so $\mu(A \setminus F) = \mu(H \setminus A^c) < \varepsilon$.

Corollary Assume $A \in \mathcal{L}$, $\mu(A) < \infty$, $\varepsilon > 0$. Then there exists a compact set $K \subseteq A$ such that $\mu(A \setminus K) < \varepsilon$.

Proof: Pick F as in part 2) of the theorem.

Set $K_n := F \cap [-n, n]$, $n \in \mathbb{N}$. Then each K_n is closed and bounded, hence compact.

Moreover, $F = \bigcup_{n \in \mathbb{N}} K_n$, and

$$\underline{\underline{A \setminus F}} = A \cap F^c = A \cap \left(\bigcap_{n \in \mathbb{N}} K_n^c\right) = \bigcap_{n \in \mathbb{N}} \left(A \cap K_n^c\right)$$

$$= \bigcap_{n \in \mathbb{N}} (A \setminus K_n)$$

Since $(A \setminus K_{n+1}) \subseteq (A \setminus K_n)$ for all n, continuity from above for μ gives that

$\underbrace{\mu(A \setminus F)}_{< \varepsilon} = \lim_{n \to \infty} \mu(A \setminus K_n)$. So we can pick $N \in \mathbb{N}$ such that $\mu(A \setminus K_N) < \varepsilon$

and set $\underline{\underline{K = K_N}}$.