Chap. 1 Normed spaces and bounded linear operators.

1.1 Preliminaries: read yourself! (cf. MAT 2400 / Spaces by Tom Lindstrøm)

1.2 Norms and seminorms

We let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$.

Let V be a vector space over $\mathbb{F}$.

A seminorm on V is a map $v \to \|v\|'$ from V into $[0,\infty)$ satisfying that
$$\begin{cases} \|\lambda v\|' = |\lambda|\,\|v\|' \\ \|v+w\| \le \|v\| + \|w\| \end{cases} \quad \text{for all } v,w \in V \text{ and } \lambda \in \mathbb{F}.$$

Note that $\underset{\substack{\uparrow \\ \text{the zero vector} \\ \text{in } V}}{\|0\|'} = \|\underset{\substack{\uparrow \\ \text{the} \\ \text{number zero}}}{0}\cdot v\|' = \underbrace{|0|}_{=0}\|v\|' = 0$ (here v is any element in V)

Moreover, note that the seminorm $\|\cdot\|'$ is a norm if and only if
$\|v\|' = 0 \Rightarrow v = 0$

Example Let X be a vector space (over $\mathbb{F}$) and Y be a normed space —"—, with norm $\|\cdot\|$.
Let $T \in \mathcal{L}(X,Y) = \{$all linear maps from X into $Y\}$
For each $x \in X$, define $\|x\|_T := \|T(x)\|$.
Then $\|\cdot\|_T$ is a seminorm on X:

F.ex. if $x, x' \in X$, then
$$\begin{aligned} \|x+x'\|_T &= \|T(x+x')\| \\ &\overset{T \text{ is linear}}{=} \|T(x)+T(x')\| \\ &\overset{\Delta\text{-inequality in } (Y, \|\cdot\|)}{\le} \underbrace{\|T(x)\|}_{=\|x\|_T} + \underbrace{\|T(x')\|}_{=\|x'\|_T} \end{aligned}$$
showing that the Δ-ineq. holds in $(X, \|\cdot\|_T)$.

Since $\|x\|_T = 0 \Leftrightarrow \|T(x)\| = 0 \underset{\|\cdot\| \text{ is a norm}}{\Leftrightarrow} T(x) = 0 \Leftrightarrow x \in \mathrm{Ker}(T)$

So $\|\cdot\|_T$ is a norm on X if and only if $\mathrm{Ker}(T) = \{0\}$
i.e. T is 1-1 (= injective)

Let's consider $X = \mathbb{R}^3$, and $Y = \mathbb{R}^2$ with Euclidean norm
i.e. $\|(y_1,y_2)\|_2 := \sqrt{y_1^2 + y_2^2}$.

Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the lin. map given by
$$T((x_1,x_2,x_3)) = (x_1,x_2).$$
Then T is not 1-1 (since $\mathrm{Ker}(T) = \{(0,0,t) : t \in \mathbb{R}\} \neq \{(0,0,0)\}$)

So $\|(x_1,x_2,x_3)\|_T = \|(x_1,x_2)\|_2$ is not a norm on $\mathbb{R}^3$, only a seminorm.

From a vector space V equipped with a seminorm $\|\cdot\|'$, we can form a normed space $(\tilde{V}, \|\cdot\|)$ as follows:

We set $v \sim w \overset{\text{def}}{\iff} \|v-w\|' = 0$ for $v, w \in V$.

Then $\sim$ is an equivalence relation on V.

(F. ex. if $v \sim w$ and $w \sim u$, then $\|v-u\|' = \|(v-w)+(w-u)\|' \leq \underbrace{\|v-w\|'}_{=0} + \underbrace{\|w-u\|'}_{=0} = 0$

so $v \sim u$)

We let $[v] := \{ w \in V \mid v \sim w \}$ denote the equivalence class of v.

and set $\tilde{V} := \{ [v] : v \in V \}$. Then we define

$$\boxed{\begin{cases} [v] + [v'] := [v+v'] & \text{for all } [v], [v'] \in \tilde{V} \\ \lambda \cdot [v] := [\lambda v] & \text{and } \lambda \in \mathbb{F} \end{cases}}$$

One checks that these operations are well-defined and that $\tilde{V}$ becomes a vector space (over $\mathbb{F}$). Moreover, we set

$$\boxed{\|[v]\| := \|v\|' \quad \text{for } [v] \in \tilde{V}}$$

Then this gives a norm on $\tilde{V}$. (F. ex., assume $\|[v]\| = 0$. Then $\|v\|' = 0$, i.e. $\|v-0\|' = 0$, i.e. $v \sim 0$, i.e. $[v] = [0]$ ← the zero vector in $\tilde{V}$.

We will use this procedure in connection with $\mathcal{L}^p$-spaces.

1.3 Aspects of finite-dimensionality

- When X is a normed space (over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$), we consider it as a metric space w.r.t. to the metric $d(x,y) := \|x-y\|$ ass. with the norm $\|\cdot\|$.

- Two norms on a vect. space X, say $\|\cdot\|_1$ and $\|\cdot\|_2$, are called <u>equivalent</u> if there exist $C, D > 0$ s.t.

$$\|x\|_2 \leq C\|x\|_1 \quad \text{and} \quad \|x\|_1 \leq D\|x\|_2 \quad \text{for all } x \in X.$$

- When $X = \mathbb{F}^n$ (or when X is finite dimensional), then it is shown in Lindstrøm's book that <u>all</u> norms are equivalent. It is also shown there that a subset of $\mathbb{F}^n$ is <u>compact</u> (w.r.t. a given norm) if and only if it is <u>closed</u> and <u>bounded</u> (This is called the Heine-Borel property). We will show that this holds in any <u>finite-dim.</u> normed space.

<u>**Lemma:**</u> Let X, Y be <u>finite-dimensional</u> normed spaces (over $\mathbb{F}$) and assume $T: X \to Y$ is an <u>isomorphism</u> of vector spaces (i.e. T is linear and bijective, in which case T^{-1} is also linear). Then <u>T is an isomorphism of normed spaces</u> (i.e. T is linear, bijective and both T and T^{-1} are bounded)

<u>Proof:</u> Let $\|\cdot\|$ denote the norm on X and $\|\cdot\|'$ — " — Y.

For $x \in X$, set $\|x\|_T := \|T(x)\|'$. Since T is 1-1, $\|\cdot\|_T$ is a norm on X. (cf. the example in 1.2)

Since X is finite-dim., $\|\cdot\|_T$ is equivalent to $\|\cdot\|$. So there exists $C > 0$ such that

$\|x\|_T \leq C\|x\|$ for all $x \in X$, i.e.

$\|T(x)\|' \leq C\|x\|$ — " —, i.e.

<u>T is bounded</u>.

Now, $T^{-1}: Y \to X$ is an isom. of vect. spaces, so we get similarly <u>that T^{-1} is bounded</u>.

Proposition Let X be a fin. dim. normed space.
Then X has the Heine-Borel property:
a subset $K \subseteq X$ is compact $\iff$ K is closed and bounded.

In particular, $X_1 := \{x \in X : \|x\| \leq 1\}$ is compact (since it is closed and bounded).

Note that if X is not fin. dim., then X_1 is not compact (exercise)

Proof: ($\Rightarrow$) holds in every metric space.

($\Leftarrow$) Let $\underset{\displaystyle X}{\overset{}{K \atop \cap}}$ be closed and bounded. We assume $X \neq \{0\}$.

Let $m = \dim X \geq 1$ and pick a basis $\mathcal{B}$ for X and let $T : X \to \mathbb{F}^m$, $x \mapsto [x]_{\mathcal{B}}$ be the coord. map w.r.t. $\mathcal{B}$.

Then T is an isomorphism of vect. spaces, so it is an isom. of normed spaces (by the lemma).

Set $K' := T(K) \subseteq \mathbb{F}^m$.

Then K' is bounded: Since K is bounded, we can find $M > 0$ s.t. $\|x\| \leq M$ for all $x \in K$. Then

$$\|T(x)\| \leq \underbrace{\|T\|}_{\text{the operator norm of } T \text{ (which is finite since } T \text{ is bounded)}} \|x\| \leq \|T\| \cdot M \quad \text{for all } x \in K$$

Moreover, K' is closed: Since T^{-1} is bounded, it is continuous.

So $K' = (\underbrace{T^{-1}}_{\text{is cont.}})^{-1}(\underbrace{K}_{\text{is closed}})$ is closed.

Using the Heine-Borel prop. for $\mathbb{F}^m$, we get that K' is compact.

Since T^{-1} is continuous, $K = \underbrace{T^{-1}}_{\text{is cont.}}(\underbrace{K'}_{\text{is comp.}})$ is compact.

We also note:

Proposition $\boxed{\text{Let } X \text{ be a finite-dim. normed space. Then } X \text{ is complete (as a metric space), i.e. } X \text{ is a Banach space.}}$

Proof. We assume $X \neq \{0\}$. Let $\{x_n\}$ be a Cauchy seq. in X.

Pick an isomorphism $T: X \to \mathbb{F}^m$ where $m = \dim(X)$.

Consider $\mathbb{F}^m$ with the Euclidean norm $\|\cdot\|_2$.

Then $\{T(x_n)\}$ is a Cauchy-seq. in $\mathbb{F}^m$

(because $\| T(x_n) - T(x_k) \|_2 = \| T(x_n - x_k)\|_2 \underset{\substack{\uparrow \\ T \text{ is bounded} \\ \text{(by the lemma)}}}{\leq} \|T\| \, \|x_n - x_k\|$)

Since $\mathbb{F}^m$ is complete, we get that

there exists some $y \in \mathbb{F}^m$ s.t. $T(x_n) \to y$ as $n \to \infty$

Set $x := T^{-1}(y) \in X$. Then we get that

$$\| x_n - x \| = \| T^{-1}(T(x_n) - y) \| \underset{\substack{\uparrow \\ T^{-1} \text{ is bounded} \\ \text{(by the lemma)}}}{\leq} \|T^{-1}\| \, \underbrace{\|T(x_n) - y\|}_{\substack{\downarrow \\ 0 \text{ as } n \to \infty}}$$

$$\to 0 \text{ as } n \to \infty$$

i.e. $\{x_n\}$ converges to x. So X is complete.