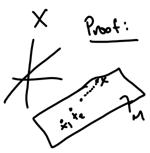
1.3 Aspects of finite-dimensionality (II)

Have seen that a finite-dim. normed space is a Banach space.

As a corollary, we get:

<u>Corollary</u>: $\boxed{\text{Assume } M \text{ is a \underline{finite-dimensional} subspace of a normed space } X. \text{ Then } M \text{ is closed in } X.}$



<u>Proof</u>: Assume $\{x_n\}$ is a sequence in M converging to $x \in X$. We have to show that $x \in M$.

Now, $\{x_n\}$, being convergent, is a Cauchy sequence.

But $(M, \|\cdot\|)$ is a Banach space since it is fin.-dim.

This means that $\{x_n\}$ converges to some $m \in M$. So we get that $x = \lim_{n\to\infty} x_n = m \in M$.

<u>Note</u>: This result does <u>not</u> hold when M is inf. dimensional (and $M \neq X$)

(Consider $X = C([a,b], \mathbb{R})$ with $\|f\| = \sup\{|f(x)|, x \in [a,b]\}$ and $M = \underbrace{\text{all real polynomial in one variable on } [a,b]}_{\text{is inf. dim.}}$

$\overline{M}^{\|\cdot\|} = X$ by Weierstrass' theorem

the closure of M w.r.t. $\|\cdot\|$)

Another useful result:

<u>Prop.</u> Let X, Y be normed spaces (over $\mathbb{F}$) and let $T : X \to Y$ be linear. Assume X is <u>finite-dimensional</u>.

Then T is bounded ($\Leftrightarrow$ continuous)

<u>Proof</u>: We first consider the case where $X = \mathbb{F}^n$, $Y = \mathbb{F}^m$.

Let $\underbrace{A = [a_{ij}]}_{m\times n \text{ matrix over } \mathbb{F}.}$ be the standard matrix of $T : \mathbb{F}^n \to \mathbb{F}^m$

For $x = (x_1, \dots, x_n)$, then $T(x) \overset{T(x) = Ax}{=} (F_1(x), \dots, F_m(x))$

where $F_i(x) = \sum_{j=1}^n a_{ij} x_j$ for $i = 1, \dots, m$

The maps $F_i : \mathbb{F}^n \to \mathbb{F}$ are clearly continuous,

So we see that T is continuous, hence bounded.

Consider now the general situation.

Set $Y' := T(X)$ and consider it as a normed space.

We may then consider T as a map from X into Y'.

Since X is fin. dim. and T is linear, Y' is also <u>fin. dim.</u>

Set $\begin{cases} n := \dim X \\ m := \dim Y' \end{cases}$

$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[d, shift right, "C"'] & Y' \arrow[d, shift left, "D"] \\
\mathbb{F}^n \arrow[u, shift right, "C^{-1}"'] \arrow[r, dashed, "S = D \circ T \circ C^{-1}"'] & \mathbb{F}^m \arrow[u, shift left, "D^{-1}"]
\end{tikzcd}$$

If $n = 0$, i.e. $X = \{0\}$, or $m = 0$, i.e. $Y' = \{0\}$, then the assertion to be proven is trivial.

So we can assume $n, m > 0$. We may then pick

$\begin{cases} C : X \to \mathbb{F}^n & \text{isomorphism of vect. spaces.} \\ D : Y \to \mathbb{F}^m & \text{—"— —"—} \end{cases}$

The map $S := D \circ T \circ C^{-1}$ is then a linear map from $\mathbb{F}^n$ to $\mathbb{F}^m$, so from above, we know that it is <u>continuous</u>.

As $T = D^{-1} \circ T \circ C$ and C and D are bounded (= continuous) (using the lemma from last lecture),

we get that T is continuous (= bounded), being the comp. of continuous maps.

<u>Note</u>: The result above does <u>not</u> hold if we instead assume Y is finite-dim, even when $Y = \mathbb{F}$. There are examples where X is inf. dim., and $T : X \to \mathbb{F}$ is linear but <u>unbounded</u> (= not cont.)

1.4 Extension by density and continuity.

This is a useful principle!

Theorem: Assume X is a normed space, Y is a Banach space, X_0 is a dense subspace of X, Y_0 is a subspace of Y.

(i.e. $\overline{X_0}^{\|\cdot\|} = X$)

Let $T_0 \in \mathcal{B}(X_0, Y_0)$, i.e. $T_0 : X_0 \to Y_0$ is linear and bounded (= continuous).

Then T_0 can be extended in a unique way to some $T \in \mathcal{B}(X, Y)$.

We have that $\|T\| = \|T_0\|$.

operator norms

We will later apply this principle to the foll. situation:

$$X_0 = C([a,b], \mathbb{C}) \quad \text{with} \quad \|f\|_2 := \left(\int_a^b |f(t)|^2 \, dt \right)^{1/2}$$

Let $K : [a,b] \times [a,b] \to \mathbb{C}$ be a continuous function.

Can then form $T_0 : X_0 \to X_0$ given by

$$[T_0(f)](s) = \int_a^b K(s,t) \, f(t) \, dt \quad , \quad s \in [a,b].$$

which is linear and bounded. But X_0 is not a Banach space.

But we will see that X_0 is dense in $\underbrace{L^2([a,b])}_{\text{the } L^2\text{-space ass with Leb. measure, which is a Banach space (even a Hilb. space)}}$

So we will be able to extend T_0 to $L^2([a,b])$, using the principle above.

We will need the foll. lemma for the proof:

Lemma Let X, Y be metric spaces, and X_0 be a dense subset of X. Assume $f, g : X \to Y$ are continuous and such that $f = g$ on X_0. Then $f = g$ on X.

Proof: Let $x \in X$. Since $\overline{X_0} = X$, we can pick $\{x_n\} \subseteq X_0$ s.t. $x_n \to x$. But then we get

$$f(x) \underset{\substack{\uparrow \\ f \text{ is cont.}}}{=} \lim_{n\to\infty} f(x_n) \underset{\substack{\uparrow \\ f=g \text{ on } X_0}}{=} \lim_{n\to\infty} g(x_n) \underset{\substack{\uparrow \\ g \text{ is cont.}}}{=} g(x).$$

Proof of the extension principle: It is clear that we can assume that $T_0 \neq 0$ (the zero map), so $\|T_0\| \neq 0$.

Let $x \in X$. Since $\overline{X_0} = X$, we can pick $\{x_n\} \subseteq X_0$ s.t. $x_n \to x$. Then $\{x_n\}$ is a Cauchy-sequence $\underbrace{}_{\text{in } X_0}$, so it follows that $\{T_0(x_n)\}$ is a Cauchy-sequence in Y_0, hence also in Y. Indeed,

Let $\varepsilon > 0$. Then choose $N \in \mathbb{N}$ s.t. $\|x_m - x_n\| < \frac{\varepsilon}{\|T_0\|}$ $\forall m, n \geq N$.

Then
$$\|T_0(x_m) - T_0(x_n)\| = \|T_0(x_m - x_n)\| \underset{\substack{\uparrow \\ T_0 \text{ is bounded}}}{\leq} \|T_0\| \underbrace{\|x_m - x_n\|}_{< \frac{\varepsilon}{\|T_0\|}} < \varepsilon$$
$\forall m, n \geq N$.

Since Y is a Banach space, $\{T_0(x_n)\}$ converges to some $y \in Y$.

Note that y only depends on x: for if $\{x_n'\}$ is another seq. in X_0 converging to x, then the sequence

$\{x_m''\}_{m\in\mathbb{N}} := \{x_1, x_1', x_2, x_2', \dots, x_n, x_n', \dots\}$ in X_0 also converges to x.

So, by the same argument as above, $\exists\, z \in Y$ s.t. $T_0(x_m'') \to z$ as $m \to \infty$.

This implies that $\lim_{n\to\infty} T_0(x_n') = z = \lim_{n\to\infty} T_0(x_n) = y$

So it makes sense to define $T(x) := y$. In this way, we get a map $T : X \to Y$, and one easily checks that T is linear.

Moreover, T is bounded:

Let $x \in X$. Pick $\{x_n\} \subseteq X_0$ s.t. $x_n \to x$.

Then $T(x) = \lim_{n\to\infty} T_0(x_n)$ and $\|T_0(x_n)\| \leq \|T_0\| \|x_n\|$ $\forall n$

$$\text{So } \|T(x)\| \underset{\substack{\uparrow \\ \text{the norm is} \\ \text{a cont. function}}}{=} \lim_{n\to\infty} \|T_0(x_n)\| \leq \lim_{n\to\infty} \|T_0\| \|x_n\| \underset{\substack{\uparrow \\ \text{cont. of} \\ \text{the norm}}}{=} \|T_0\| \|x\|$$

which shows that T is bounded, with $\|T\| \leq \|T_0\|$.

T is an extension of T_0: Let $x \in X_0$. Set $x_n = x$ $\forall n \in \mathbb{N}$.

Then $T(x) = \lim_{n\to\infty} \underbrace{T_0(x_n)}_{T_0(x)} = T_0(x)$.

The lemma gives that T is the only extension of T_0 to X.

Finally,
$$\begin{aligned}\|T_0\| &= \sup\{\|T_0(x)\| : x \in X_0, \|x\| \leq 1\} \\ &= \sup\{\|T(x)\| : \text{— '' —}\} \\ &\leq \sup\{\|T(x)\| : x \in X, \|x\| \leq 1\} = \|T\|,\end{aligned}$$

So we get $\|T_0\| \leq \|T\| \leq \|T_0\|$, i.e. $\|T\| = \|T_0\|$.
