1.4 The extension principle (II)

Recall:

$$\begin{array}{lccccl} \text{normed space} & X & \overset{T \ \text{bounded}}{\dashrightarrow} & Y & \text{Banach space} \\ & \cup\!\!| & \text{extends to} & \cup\!\!| & \\ \text{dense subspace} & X_0 & \xrightarrow[\text{bounded}]{T_0} & Y_0 & \text{subspace} \end{array}$$

Have that $T(x) = \lim_{n\to\infty} T_0(x_n)$ where $\{x_n\} \subseteq X_0$ such that $x_n \to x$

and that $\|T\| = \|T_0\|$.

Corollary: Assume that T_0 above is an isometry, that is $\|T_0(x)\| = \|x\|$ for every $x \in X_0$. Then T is an isometry from X to Y.

Proof Indeed, let $x \in X$ and pick $\{x_n\} \subseteq X_0$ s.t. $x_n \to x$.

Then $T_0(x_n) \to T(x)$ as $n\to\infty$, so $\underbrace{\|T_0(x_n)\|}_{\|x_n\|} \to \|T(x)\|$ as $n\to\infty$

$\|x_n\| \to \|x\|$ as $n\to\infty$.

Hence $\|T(x)\| = \|x\|$.

If X is a normed space, we set $\mathcal{B}(X) := \mathcal{B}(X,X)$
$= \{T : X \to X \mid T \text{ linear and bounded}\}$

Corollary Assume X is a Banach space, X_0 is a dense subspace of X, and $T_0 \in \mathcal{B}(X_0)$.

Then T_0 has a unique extension to $T \in \mathcal{B}(X)$, satisfying $\|T\| = \|T_0\|$.

Proof: We apply the ext. principle with $Y = X$, $Y_0 = X_0$.

Chap. 2 L^p-spaces

2.1 The case $p \in [1,\infty)$.

$(X, \mathcal{A}, \mu)$ measure space

$\mathcal{M} := \{ f : X \to \mathbb{C} \mid f \text{ is measurable (w.r.t. } \mathcal{A}) \}$ vect. space (w.r.t. usual operations)

$p \in [1,\infty)$. We set

$$\underbrace{\mathcal{L}^p(X,\mathcal{A},\mu)}_{\text{we will usually write just } \mathcal{L}^p.} := \left\{ f \in \mathcal{M} : \int_X |f|^p \, d\mu < \infty \right\}.$$

Note that $\mathcal{L}^1 = \{ f \in \mathcal{M} : f \text{ is integrable w.r.t. } \mu \}$

Note also that $\boxed{\mathcal{L}^p \text{ is a subspace of } \mathcal{M}}$:

Let $f, g \in \mathcal{L}^p$. Then $f+g \in \mathcal{M}$ and

$$|f+g|^p \le (|f|+|g|)^p \le (2\max(|f|,|g|))^p$$
$$\le 2^p (|f|^p + |g|^p), \text{ so}$$

$$\int_X |f+g|^p \, d\mu \le 2^p \Big(\underbrace{\int_X |f|^p \, d\mu}_{<\infty} + \underbrace{\int_X |g|^p \, d\mu}_{<\infty} \Big) < \infty,$$

i.e. $f+g \in \mathcal{L}^p$.

If $\lambda \in \mathbb{C}$, then $\lambda f \in \mathcal{L}^p$ (easy).

For $f \in \mathcal{L}^p$, we set $\boxed{\|f\|_p := \left(\int_X |f|^p \, d\mu \right)^{1/p}}$

Note that

$\|f\|_p = 0 \iff \int_X |f|^p \, d\mu = 0 \iff |f|^p = 0$ μ-a.e. $\iff f = 0$ μ-a.e.

(so $\|\cdot\|_p$ is not a norm. We are going to show that it is a seminorm)

Example Consider $X = [1,\infty)$, $\mathcal{A}$ = Leb. measurable subsets of X and μ = Leb. measure.

Let $p \geq 1$. Consider $f(x) = \frac{1}{x}$, $x \in X$.

One easily computes that $\int_{[1,\infty)} |f|^p \, d\mu = \int_{[1,\infty)} \frac{1}{x^p} \, d\mu(x) < \infty$

$$\Leftrightarrow \underline{p > 1}.$$

So $\underline{f \in \mathcal{L}^p \Leftrightarrow p > 1}$.

For $r > 0$, set $f_r(x) = \frac{1}{x^r}$, $x \in X$.

Similarly, we get $f_r \in \mathcal{L}^p \Leftrightarrow rp > 1$
$\Leftrightarrow r > \frac{1}{p}$.

So f.ex. $\begin{cases} f_{3/4} \in \mathcal{L}^2 & (\text{since } \frac{3}{4} > \frac{1}{2}) \\ f_{1/3} \notin \mathcal{L}^2 & (\text{since } \frac{1}{3} \not> \frac{1}{2}) \end{cases}$

Example $X \neq \emptyset$ set, $\mathcal{A} = \mathcal{P}(X)$, μ = counting measure.

$p \geq 1$.

$\mathcal{L}^p(X, \mathcal{P}(X), \mu)$ is denoted by $\underline{\ell^p(X)}$.

Note that if $g : X \to [0,\infty)$, then

$$\int_X g \, d\mu = \sum_{x \in X} g(x) \quad (\text{check!})$$

So if $f \in \mathcal{L}^p(X)$, then $\|f\|_p = \left(\sum_{x \in X} |f(x)|^p \right)^{1/p}$.

Set $\ell^p := \ell^p(\mathbb{N})$.

Then if $f \in \ell^p$, then $\|f\|_p = \left(\sum_{n=1}^{\infty} |f(n)|^p \right)^{1/p}$

It is then usual to consider elements in ℓ^p as sequences (via the map $f \leftrightarrow \underbrace{\{f(n)\}}_{=:\{x_n\}}$), with

$$\|\{x_n\}\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}.$$

Def. Let $p \in (1,\infty)$. Then the conjugate exponent of p is defined by $\boxed{q := \frac{p}{p-1} \in (1,\infty)}$.

(One easily sees that $\frac{1}{p} + \frac{1}{q} = 1$)

Theorem (Hölder's inequality)

Let $p \in (1,\infty)$ and q be the conj. exp. of p.
Let $f \in \mathcal{L}^p$ and $g \in \mathcal{L}^q$. Then $fg \in \mathcal{L}^1$

and $\|fg\|_1 = \int_X |fg| \, d\mu \leq \|f\|_p \|g\|_q$

The proof uses Young's inequality: with p, q as above.

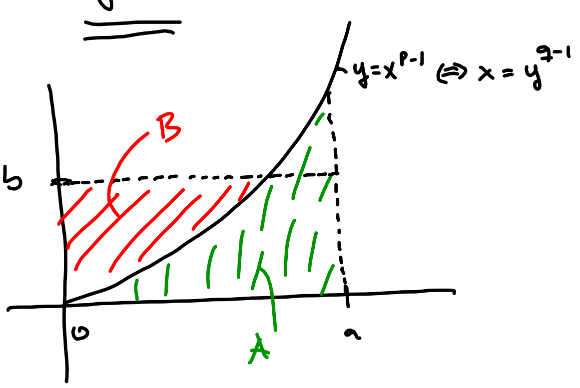
let $a, b > 0$. Then $\boxed{ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q}$

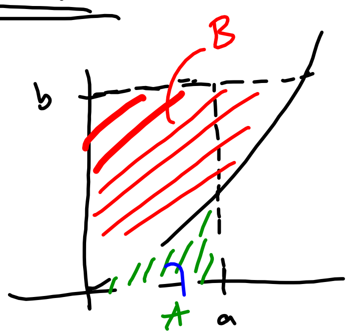
Note that
$$\begin{cases} \frac{1}{p} a^p = \int_0^a x^{p-1}\,dx = \underbrace{\text{area under } y = x^{p-1} \text{ over } [0,a]}_{A} \\ \frac{1}{q} b^q = \int_0^b y^{q-1}\,dy = \underbrace{\text{area under } x = y^{q-1} \text{ over } [0,b]}_{B} \end{cases}$$

Note that $q - 1 = \frac{p}{p-1} - \frac{p-1}{p-1} = \frac{1}{p-1}$, so we have

$y = x^{p-1} \iff x = y^{1/(p-1)} \iff x = y^{q-1}$ $(x, y \geq 0)$

$a \geq b$





$a \leq b$

In both cases, we see that $ab \leq A + B = \frac{1}{p} a^p + \frac{1}{q} b^q$.

We can now prove Hölder's inequality.

- If $\|f\|_p = 0$ or $\|g\|_q = 0$, then $f=0$ μ-a.e or $g=0$ μ-a.e then the inequality is ok.

- Assume $\|f\|_p = 1 = \|g\|_q$. Then

$$\int_X |fg|\,d\mu = \int_X |f(x)|\,|g(x)|\,d\mu(x)$$

$$\underset{\text{Young's ineq.}}{\leq} \int_X \left(\frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q\right) d\mu(x)$$

$$= \frac{1}{p}\underbrace{\|f\|_p^p}_{=1} + \frac{1}{q}\underbrace{\|g\|_q^q}_{=1} = \frac{1}{p} + \frac{1}{q} = 1$$

$$= \underbrace{\|f\|_p}_{=1}\underbrace{\|g\|_q}_{=1}$$

which shows the ineq. in this case.

- Finally, assume $\|f\|_p \neq 0$, $\|g\|_q \neq 0$.

Set $f_1 = \frac{1}{\|f\|_p} f$, $g_1 = \frac{1}{\|g\|_q} g$.

Then $\|f_1\|_p = 1 = \|g_1\|_q$, so we get that

$$\underbrace{\int_X |f_1 g_1|\,d\mu}_{= \frac{1}{\|f\|_p \|g\|_q}\int_X |fg|\,d\mu} \leq \underbrace{\|f_1\|_p}_{=1}\underbrace{\|g_1\|_q}_{=1} = 1$$

which gives Hölder's ineq.

Corollary Let $p \in [1,\infty)$. Then $\|\cdot\|_p$ is a seminorm on $\mathcal{L}^p$.
In particular, $\|f+g\|_p \le \|f\|_p + \|g\|_p \quad \forall f,g \in \mathcal{L}^p$.
(This is called Minkowski's inequality)

Proof If $p=1$, then this follows immediately from the usual Δ-inequality.

So we assume $p>1$, and let q be the conj. exponent of p.

Note that $\boxed{(p-1)q = p, \text{ so } \frac{p}{q} = p-1.}$

So we get

$$\| |f+g|^{p-1} \|_q = \left(\int_X |f+g|^{(p-1)q} \, d\mu \right)^{1/q} = \left(\int_X |f+g|^p \, d\mu \right)^{1/q} = \|f+g\|_p^{p/q} = \|f+g\|_p^{p-1} < \infty$$

since $f,g \in \mathcal{L}^p$

This means that $|f+g|^{p-1} \in \mathcal{L}^q$. Thus we get

$$\|f+g\|_p^p = \int_X |f+g|^p \, d\mu = \int_X \underbrace{|f+g|}_{\le |f|+|g|} |f+g|^{p-1} \, d\mu$$

$$\le \int \underbrace{|f|}_{\text{is in } \mathcal{L}^p} \underbrace{|f+g|^{p-1}}_{\text{is in } \mathcal{L}^q} d\mu + \int \underbrace{|g|}_{\text{is in } \mathcal{L}^p} \underbrace{|f+g|^{p-1}}_{\text{is in } \mathcal{L}^q} d\mu$$

Hölder's ineq. $\leadsto$

$$\le \|f\|_p \, \| |f+g|^{p-1} \|_q + \|g\|_p \, \| |f+g|^{p-1} \|_q$$

$$= (\|f\|_p + \|g\|_p) \underbrace{\| |f+g|^{p-1} \|_q}_{= \|f+g\|_p^{p-1}}$$

as computed above!

If $\|f+g\|_p = 0$, then Minkowski's ineq. is trivially true.

If $\|f+g\|_p \neq 0$, we can divide by $\|f+g\|_p^{p-1}$ and get it right!

Homogeneity of $\|\cdot\|_p$ is easy, so $\|\cdot\|_p$ is a seminorm.

We will need (next time)

LDCT ($\mathcal{L}^p$-version): Assume $\{f_n\} \subseteq \mathcal{L}^p$, $f_n \to f$ pointwise μ-a.e., $f \in \mathcal{M}$, $|f_n| \le g$ μ-a.e., $g \in \mathcal{L}^p$.

Then $f \in \mathcal{L}^p$ and $\|f_n - f\|_p \to 0$ as $n \to \infty$

Proof. We have $|f_n|^p \le g^p$ μ-a.e.

$|f_n|^p \to |f|^p$ pointwise μ-a.e.

So LDCT gives that $\int_X |f|^p \, d\mu = \lim_{n\to\infty} \int_X |f_n|^p \, d\mu \le \int_X g^p \, d\mu < \infty$

so $f \in \mathcal{L}^p$.

Moreover, $|f_n - f|^p \le (|f_n| + |f|)^p \le \underbrace{2^p g^p}_{\text{is integrable}}$

and $|f_n - f|^p \to 0$ pointwise μ-a.e.

So LDCT gives that

$$\lim_{n\to\infty} \underbrace{\int_X |f_n - f|^p \, d\mu}_{\|f_n - f\|_p^p} = \int_X \underbrace{\lim_{n\to\infty} |f_n - f|^p}_{= 0} \, d\mu = 0$$