L^p- spaces II ($p\in[1,\infty)$)

$(X, \mathcal{A}, \mu)$ measure space , $p \in [1,\infty)$

$\mathcal{L}^p = \{ f: X \to \mathbb{C} \mid f \text{ is measurable }, \|f\|_p < \infty \}$

where $\|f\|_p := \left(\int_X |f|^p \, d\mu \right)^{1/p}$

Have seen that $\|\cdot\|_p$ is a seminorm on $\mathcal{L}^p$.

We set $f \sim g \Leftrightarrow \|f-g\|_p = 0$ $(\Leftrightarrow f = g$ μ-a.e$)$
for $f, g \in \mathcal{L}^p$. Moreover, we set

$[f] := \{ g \in \mathcal{L}^p : f \sim g \} = \{ g \in \mathcal{L}^p : f = g \ \mu\text{-a.e.} \}$

Then we get that

$\underbrace{L^p(X, \mathcal{A}, \mu)}_{L^p} := \{ [f] : f \in \mathcal{L}^p \}$

is a normed space w.r.t. $\|\cdot\|_p$ given by
$\|[f]\|_p := \|f\|_p$ for $f \in \mathcal{L}^p$, the operations being
given by $[f] + [g] = [f+g]$, $\lambda [f] = [\lambda f]$ for $f, g \in \mathcal{L}^p$, $\lambda \in \mathbb{C}$

Note: After a while, it is usual to just write f instead of $[f]$ and say that we identify functions in L^p which agree μ-a.e.

Theorem $\boxed{(L^p, \|\cdot\|_p) \text{ is a Banach space}}$

To show this, we will use the fact a normed space X is a Banach space $\Leftrightarrow$ [every absolutely convergent series is convergent, i.e. $\sum_{n=1}^{\infty} x_n$ is convergent in X whenever $\sum_{n=1}^{\infty} \|x_n\| < \infty$.

(cf. Spaces Prop. 5.2.3)

Proof: We consider $\{[f_n]\}_{n\in\mathbb{N}} \subseteq L^p$ such that

$\sum_{n=1}^{\infty} \|[f_n]\|_p < \infty$, i.e. $\underbrace{\sum_{n=1}^{\infty} \|f_n\|_p}_{=: S} < \infty$.

We have to show that

$\sum_{n=1}^{\infty} [f_n]$ is convergent in L^p (w.r.t. $\|\cdot\|_p$).

For each $N \in \mathbb{N}$, set $g_N := \sum_{n=1}^{N} |f_n|$, $g := \sum_{n=1}^{\infty} |f_n|$

are measurable

takes values in $[0,\infty]$

Then we have $(g_N)^p \nearrow g^p$ as $N \to \infty$, and

$$\|g_N\|_p \underset{\substack{\uparrow \\ \text{Minkowski's} \\ \text{ineq.}}}{\leq} \sum_{n=1}^{\infty} \| |f_n| \|_p = \sum_{n=1}^{N} \|f_n\|_p \leq S < \infty \quad \underline{\forall N \in \mathbb{N}}.$$

Hence we get

$$\int_X g^p \, d\mu \underset{\substack{\uparrow \\ \text{MCT}}}{=} \lim_{N\to\infty} \underbrace{\int_X (g_N)^p \, d\mu}_{= \|g_N\|_p^p} \leq S^p < \infty \text{, i.e. } \underline{g \in \mathcal{L}^p}$$

Since $g^p \geq 0$, this implies that g^p is finite μ-a.e.
i.e. $\underline{g \text{ is finite } \mu\text{-a.e.}}$

This means that there exists $E \in \mathcal{A}$ s.t. $\mu(E^c) = 0$

and $g(x) = \sum_{n=1}^{\infty} |f_n(x)| < \infty$ for all $x \in E$.

As $\mathbb{C}$ is a Banach space, we get that that

$\sum_{n=1}^{\infty} f_n(x)$ is convergent <u>for every $x \in E$</u>. So we may

define $F \in \mathcal{M}$ by
$$\boxed{F(x) = \begin{cases} \sum_{n=1}^{\infty} f_n(x) & , x \in E \\ 0 & , x \in E^c \end{cases}}$$

Set now $F_N = \sum_{n=1}^{N} f_n$, $N \in \mathbb{N}$, so $\underline{F_N \in \mathcal{L}^p}$ $\forall N \in \mathbb{N}$.

Then $|F_N| \leq \sum_{n=1}^{N} |f_n| = g_N \leq g \in \mathcal{L}^p$ $\forall N \in \mathbb{N}$.

Moreover $F_N(x) \to F(x)$ for all $x \in E$.

Hence $\underline{F_N \to F \text{ pointwise } \mu\text{-a.e.}}$

The $\mathcal{L}^p$-version of LDCT gives that $F \in \mathcal{L}^p$

and $\|F_N - F\|_p \to 0$ as $N \to \infty$.

Hence $\|[F_N] - [F]\|_p = \|[F_N - F]\|_p = \|F_N - F\|_p \to 0$ as $N \to \infty$

i.e. $\left\| \sum_{n=1}^{N} [f_n] - [F] \right\|_p \to 0$ as $N \to \infty$, as desired.

Note: It is not always true that a sequence in $\mathcal{L}^p$ wich is convergent w.r.t. $\|\cdot\|_p$ will converge pointwisely to the same function μ-a.e. However, it can be shown that:

cf. Prop. 2.1.4

If $\{h_k\}_{k\in\mathbb{N}} \subseteq \mathcal{L}^p$ converges to $h \in \mathcal{L}^p$ w.r.t. $\|\cdot\|_p$, then there exists a subsequence $\{h_{k_n}\}_{n\in\mathbb{N}}$ such that $h_{k_n} \to h$ pointwisely μ-a.e.

Read the proof if you want. This result will not be included in the final curriculum.

Some approximation results

Consider $\mathcal{E} := \text{span}\{1_A : A \in \mathcal{A}\}$ (= the simple measurable functions on X taking complex values)

What is $\mathcal{E} \cap \mathcal{L}^p$?

Prop. $p \in [1,\infty)$

Set $\mathcal{E}^0 := \text{span}\{1_A : A \in \mathcal{A}, \mu(A) < \infty\} \subseteq \mathcal{E}$.
Then $\mathcal{E}^0 = \mathcal{E} \cap \mathcal{L}^p$. Moreover,
$[\mathcal{E}^0] := \{[f] : f \in \mathcal{E}^0\}$ is dense in L^p w.r.t. $\|\cdot\|_p$.

Proof: Let $g \in \mathcal{E}$, $g \neq 0$.
Write $g = \sum_{j=1}^n \lambda_j 1_{A_j}$ (standard form) with $\lambda_j \neq 0$ for $j = 1, \dots, n$.

Then $|g|^p = \sum_{j=1}^n |\lambda_j|^p 1_{A_j}$, so

$$\int_X |g|^p \, d\mu = \sum_{j=1}^n |\lambda_j|^p \mu(A_j).$$

Hence, if $g \in \mathcal{L}^p$, then the sum above is finite, so $\mu(A_j) < \infty$ for all $j = 1, \dots, n$, i.e., $g \in \mathcal{E}^0$

This shows that $\mathcal{E} \cap \mathcal{L}^p \subseteq \mathcal{E}^0$.

Conversely, note that if $A \in \mathcal{A}$, $\mu(A) < \infty$, then $1_A \in \mathcal{L}^p$.
Since $\mathcal{L}^p$ is a vect. space, we get that $\mathcal{E}^0 \subseteq \mathcal{L}^p$, hence that $\mathcal{E}^0 \subseteq \mathcal{E} \cap \mathcal{L}^p$.

Let now $f \in \mathcal{L}^p$. Set $f_1 := \text{Re}(f)$, $f_2 := \text{Im}(f)$
so $f_1 = f_1^+ - f_1^-$, $f_2 = f_2^+ - f_2^-$.
We can then pick $\{g_k\}, \{h_k\}, \{r_k\}, \{s_k\} \subseteq \{g \in \mathcal{E} : g \geq 0\}$
s.t. $g_k \nearrow f_1^+$, $h_k \nearrow f_1^-$, $r_k \nearrow f_2^+$, $s_k \nearrow f_2^-$

Set $f_k := (g_k - h_k) + i(r_k - s_k) \in \mathcal{E}$.

Then $f_k \to f$ pointwise on X and $|f_k| \leq \underbrace{2|f|}_{\in \mathcal{L}^p}$ for all k.

As $\int_X |f_k|^p \, d\mu \leq 2^p \int_X |f|^p \, d\mu < \infty$ for each k,

we have $f_k \in \mathcal{L}^p$ for each k.

So $f_k \in \mathcal{E} \cap \mathcal{L}^p = \mathcal{E}^0$ for each k.

The $\mathcal{L}^p$-version of LDCT gives that $\|f_k - f\|_p \to 0$ as $k \to \infty$

hence that $\|\underbrace{[f_k]}_{\in [\mathcal{E}^0]} - [f]\|_p = \|f_k - f\|_p \to 0$ as $k \to \infty$

as desired.

Using this approximation result, we can deduce (proceeding in the same way as in Brevig's notes) the following:

Theorem ($1 \le p < \infty$) Let $X = [a,b]$, $\mathcal{A}$ = Leb. meas. subsets of X, μ = Leb. meas.,

$C([a,b]) = \{ f : [a,b] \to \mathbb{C} : f \text{ is continuous} \}$.

Then $\{ [f] : f \in C([a,b]) \}$ is dense in $L^p([a,b], \mathcal{A}, \mu)$

Theorem ($1 \le p < \infty$) Let $X = \mathbb{R}$, $\mathcal{A}$ = Leb. meas. subsets, μ = Leb. meas.

$C_c(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{C} : f \text{ is continuous and } f = 0 \text{ outside } [-M, M] \text{ for some } M > 0 \}$

Then $\{ [f] : f \in C_c(\mathbb{R}) \}$ is dense in $L^p(\mathbb{R}, \mathcal{A}, \mu)$

Let's now go back to integral operators.

Set $X_0 = C([a,b])$, which is a normed space w.r.t. $\|f\|_2 = \left(\int_a^b |f(t)|^2 \, dt \right)^{1/2}$

Let $K : [a,b] \times [a,b] \to \mathbb{C}$ be continuous.

We can identify X_0 with $\{ [f] : f \in C([a,b]) \}$ in $L^2([a,b])$ by the map $f \leftrightarrow [f]$. We can def. $T_K : X_0 \to X_0$ by

$$T_K(f)(s) = \int_a^b \underbrace{K(s,t) f(t)}_{\substack{\text{is a cont. function of } t \\ \text{for each } s \in [a,b]}} \, dt$$

Then T_K is linear and

$$\|T_K(f)\|_2^2 = \int_a^b |T_K(f)(s)|^2 \, ds = \int_a^b \Big| \int_a^b \underbrace{K(s,t)}_{k_s(t)} f(t) \, dt \Big|^2 ds$$

$$\le \int_a^b \|k_s\|_2^2 \, \|f\|_2^2 \, ds$$

Hölder ineq.

$$= \left(\int_a^b \int_a^b |K(s,t)|^2 \, dt \, ds \right) \|f\|_2^2$$

for all $f \in C([a,b])$,

i.e. $\|T_K(f)\|_2 \le \left(\int_a^b \int_a^b |K(s,t)|^2 \, dt \, ds \right)^{1/2} \|f\|_2 \quad \forall f$

Showing that T_K is bounded.

Using the extension principle, we can extend T_K to a bounded operator $T_K : L^2([a,b]) \to L^2([a,b])$

(because L^2 is a Banach space and $C([a,b])$ is dense in L^2)