More on L^p-spaces:

2.2 The case $p=\infty$

$(X, \mathcal{A}, \mu)$ measure space.

Set $\mathcal{F}(X) := \{ f : X \to \mathbb{C} \}$ ← is a vect. space w.r.t. its pointwise operations

A subset $\mathcal{G}$ of $\mathcal{F}(X)$ is called an algebra of (complex) functions on X when $\mathcal{G}$ is a subspace of $\mathcal{F}(X)$ which is also closed under pointwise multiplication.

F.ex. $\mathcal{M} = \mathcal{M}(X, \mathcal{A}, \mu) := \{ f \in \mathcal{F}(X) : f \text{ is measurable (w.r.t. } \mathcal{A})\}$ is an algebra of functions on X.

Another example is $\ell^\infty(X) := \{ f \in \mathcal{F}(X) : f \text{ is bounded} \}$

Hence $\mathcal{M} \cap \ell^\infty(X) = \{ f \in \mathcal{F}(X) : f \text{ is measurable and bounded}\}$ is also an alg. of functions.

It is contained in a larger algebra of functions on X:

Def We say $f \in \mathcal{M}$ is essentially bounded (w.r.t. μ) if $\exists\, M > 0$ such that $|f| \le M$ μ-a.e.

i.e. $\mu(\{x \in X : |f(x)| > M\}) = 0$,

in which case we set

$$\boxed{\|f\|_\infty := \inf\{ M > 0 : |f| \le M \ \mu\text{-a.e.}\}}$$

We set $\underbrace{\mathcal{L}^\infty(X, \mathcal{A}, \mu)}_{\text{we just write } \mathcal{L}^\infty \text{ when no confusion is possible.}} := \{ f \in \mathcal{M} : f \text{ is ess. bounded (w.r.t. } \mu)\}$

Ex. Let $g \in \mathcal{M} \cap \ell^\infty(X)$ and set $\|g\|_u := \sup_{x \in X} |g(x)|$.

Then $|g| \le \|g\|_u$ on X, so $|g| \le \|g\|_u$ (μ-a.e.)

Hence g is essentially bounded (w.r.t. μ) and $\|g\|_\infty \le \|g\|_u$.

So $\mathcal{M} \cap \ell^\infty(X) \subseteq \mathcal{L}^\infty(X, \mathcal{A}, \mu)$.

Note that it may happen that $\|g\|_\infty < \|g\|_u$:

Indeed, let $A \in \mathcal{A}$, $A \neq \emptyset$, $\mu(A) = 0$, and set $g = 1_A$.

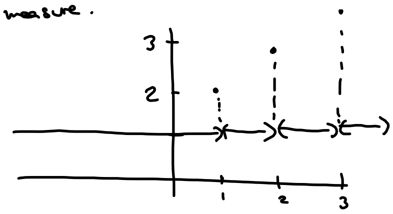
Then $\|g\|_u = 1$, but $\|g\|_\infty = 0$.

(Indeed, if $M > 0$, then $\mu(\{x \in X : |g(x)| > M\}) = \begin{cases} \mu(A) & \text{if } 0 < M < 1 \\ \mu(\emptyset) & \text{if } M \ge 1 \end{cases} = 0$

so $\|g\|_\infty = \inf\{M > 0\} = 0$.)

Ex $X = \mathbb{R}$, $\mathcal{A} = \mathcal{B}_{\mathbb{R}}$, μ = Leb. measure.

Consider $f = 1_X + \sum_{n=1}^{\infty} n\, 1_{\{n\}}$



Then f is unbounded,

but it is essentially bounded

with $\|f\|_\infty = 1$ (Indeed, let $M > 0$.

Then $\{x \in X \mid |f(x)| > M\} = \begin{cases} \mathbb{R} & \text{if } 0 < M < 1 \\ \{k, k+1, \dots\} & \text{if } M \geq 1 \end{cases}$ and $k := [M]$

So $\mu(\{x \in X \mid |f(x)| > M\}) = \begin{cases} \infty & \text{if } M < 1 \\ 0 & \text{if } M \geq 1 \end{cases}$

Thus $|f| \leq M$ μ-a.e. $\iff M \geq 1$

Hence $\|f\|_\infty = \inf\{M > 0 \mid M \geq 1\} = 1$.

Note: It can be shown that $f \in \mathcal{M}$ is ess. bounded (w.r.t. μ)

$\iff$ there exists some $g \in \mathcal{M} \cap \ell^\infty(X)$ such that

$f = g$ μ-a.e.

(Exercise!).

We will use repeatedly the following simple observation:

Lemma $\boxed{\text{Assume } f \in \mathcal{L}^\infty. \text{ Then } |f| \leq \|f\|_\infty \ \mu\text{-a.e.}}$

Proof Set $B = \{x \in X : |f(x)| > \|f\|_\infty\} \in \mathcal{A}$.

Assume (for contradiction) that $\mu(B) > 0$.

For each $n \in \mathbb{N}$, let $B_n := \{x \in X : |f(x)| > \|f\|_\infty + \frac{1}{n}\} \in \mathcal{A}$

Then $B_n \subseteq B_{n+1}$, and $B = \bigcup_{n=1}^{\infty} B_n$.

So by cont. of μ, we get $\underbrace{\mu(B)}_{>0} = \lim_{n \to \infty} \mu(B_n)$

Hence we can choose $N \in \mathbb{N}$ s.t. $\mu(B_N) > 0$.

By def. of $\|f\|_\infty$ we can find $M > 0$ s.t.

$\|f\|_\infty \leq M < \|f\|_\infty + \frac{1}{N}$ and $|f| \leq M$ μ-a.e.

So we get that $|f| \leq \|f\|_\infty + \frac{1}{N}$ μ-a.e.

i.e. $\mu(B_N) = 0$, giving a contradiction!

Thus $\mu(B) = 0$, i.e. $|f| \leq \|f\|_\infty$ μ-a.e.

Proposition: $\boxed{\mathcal{L}^\infty = \mathcal{L}^\infty(X,\mathcal{A},\mu) \text{ is an algebra of functions on } X \text{ and } \|\cdot\|_\infty \text{ is a seminorm on } \mathcal{L}^\infty \text{ s.t. } \|fg\|_\infty \le \|f\|_\infty \|g\|_\infty}$

Proof Let $f, g \in \mathcal{L}^\infty$. Then, using the previous lemma, we get

$$|f+g| \le |f| + |g| \le \|f\|_\infty + \|g\|_\infty \quad \mu\text{-a.e}$$

Hence $f+g \in \mathcal{L}^\infty$ and $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$.

We leave the rest of the proof as an exercise.

A Hölder-type result:

Proposition: $\boxed{\text{Let } q \in [1,\infty),\ f \in \mathcal{L}^\infty,\ g \in \mathcal{L}^q. \text{ Then } fg \in \mathcal{L}^q \text{ and } \|fg\|_q \le \|f\|_\infty \|g\|_q}$

Proof: We have $|fg|^q = |f|^q |g|^q \underset{\substack{\uparrow \\ \text{by the prev. lemma}}}{\le} (\|f\|_\infty)^q |g|^q$ μ-a.e

So
$$\underbrace{\int_X |fg|^q \, d\mu}_{\|fg\|_q^q} \le \int_X (\|f\|_\infty)^q |g|^q \, d\mu = (\|f\|_\infty)^q \underbrace{\int_X |g|^q \, d\mu}_{\|g\|_q^q} < \infty$$

which shows the claim.

Convergence in $\mathcal{L}^\infty$ is "almost uniform"-convergence:

Prop. Assume $\{f_n\}$ is a sequence in $\mathcal{L}^\infty$, and $f \in \mathcal{L}^\infty$.

Then $\boxed{\|f_n - f\|_\infty \to 0 \text{ as } n \to \infty} \iff \boxed{\text{There exists } E \in \mathcal{A} \text{ such that } \mu(E^c) = 0 \text{ and } f_n \to f \text{ uniformly on } E}$

Proof: $(\Rightarrow)$ Assume $\|f_n - f\|_\infty \to 0$ as $n \to \infty$.

For $n \in \mathbb{N}$, set $F_n = \{x \in X \mid |f_n(x) - f(x)| > \|f_n - f\|_\infty\} \in \mathcal{A}$

Then set $F = \bigcup_{n=1}^\infty F_n \in \mathcal{A}$.

Since $f_n - f \in \mathcal{L}^\infty$, we have $\mu(F_n) = 0$ (using the lemma!)

so $\mu(F) \le \sum_{n=1}^\infty \mu(F_n) = 0$, hence $\underline{\mu(F) = 0}$

Set $E = F^c \in \mathcal{A}$. Then $\underline{\mu(E^c) = \mu(F) = 0}$ and

$$E = \left(\bigcup_{n=1}^\infty F_n\right)^c = \bigcap_{n=1}^\infty F_n^c = \{x \in X : |f_n(x) - f(x)| \le \|f_n - f\|_\infty \text{ for all } n\}$$

Thus, if $\varepsilon > 0$, we may choose $N \in \mathbb{N}$ such that

$\|f_n - f\|_\infty < \varepsilon$ for all $n \ge N$, and we then get that

$|f_n(x) - f(x)| < \varepsilon$ for all $x \in E$ and all $n \ge N$.

This shows that $\underline{f_n \to f \text{ uniformly on } E}$.

$(\Leftarrow)$ goes along the same lines $\rightsquigarrow$ Exercise!

Note that if $f \in \mathcal{L}^\infty$, then $\boxed{\|f\|_\infty = 0 \iff f = 0 \ \mu\text{-a.e.}}$

(If $\|f\|_\infty = 0$, then the lemma gives that $|f| \leq 0$ μ-a.e., i.e. $f = 0$ μ-a.e.)
(If $f = 0$ μ-a.e., then $|f| \leq M$ μ-a.e. for all $M > 0$, hence $\|f\|_\infty = 0$ ✓

We can now proceed as in the case where $p \in [1, \infty)$:
We identify functions in $\mathcal{L}^\infty$ which agree μ-a.e, and get a normed space $\underbrace{L^\infty(X, \mathcal{A}, \mu)}_{L^\infty} = \{ [f] : f \in \mathcal{L}^\infty(X, \mathcal{A}, \mu) \}$

where $[f] = \{ g \in \mathcal{L}^\infty : g = f \ \mu\text{-a.e.} \}$

with the norm $\|[f]\|_\infty := \|f\|_\infty$.

Theorem $\boxed{(L^\infty, \|\cdot\|_\infty) \text{ is a Banach space}}$

Proof: Let $\{[f_n]\}$ be a Cauchy-sequence in L^∞ (wrt. $\|\cdot\|_\infty$).

For $m, n \in \mathbb{N}$, set $F_{m,n} := \{ x \in X \mid |f_m(x) - f_n(x)| > \|f_m - f_n\|_\infty \} \in \mathcal{A}$

Since $f_m - f_n \in \mathcal{L}^\infty$, we have $\mu(F_{m,n}) = 0$ (using the lemma!)

So setting $F = \bigcup_{m,n \in \mathbb{N}} F_{m,n} \in \mathcal{A}$, we get $\mu(F) = 0$

Set $E := F^c \in \mathcal{A}$. Then $\mu(E^c) = \mu(F) = 0$. Moreover,

$E = \bigcap_{m,n \in \mathbb{N}} (F_{m,n})^c = \{ x \in X \mid |f_m(x) - f_n(x)| \leq \|f_m - f_n\|_\infty \text{ for all } m, n \}$

Now, let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that

$\|f_m - f_n\|_\infty = \|[f_m] - [f_n]\|_\infty < \varepsilon$ for all $m, n \geq N$.

Then for every $x \in E$, we have

$|f_m(x) - f_n(x)| < \varepsilon$ for all $m, n \geq N$.

This means that $\{f_m(x)\}$ is a Cauchy-seq. in $\mathbb{C}$ for every $x \in E$. Since $\mathbb{C}$ is complete, we get $\{f_m(x)\}$ is convergent for every $x \in E$.

Hence $f_m(x) \to g(x)$ for some $g(x) \in \mathbb{C}$ for every $x \in E$.

Define now $f : X \to \mathbb{C}$ by

$$f(x) = \begin{cases} g(x), & x \in E \\ 0, & x \notin E. \end{cases}$$

Then $f \in \mathcal{L}^\infty$, and $f_n \to f$ uniformly on E

(so that $\|[f_n] - [f]\|_\infty = \|f_n - f\|_\infty \to 0$ by using the prev. conv.)

see the notes!