Chapter 3 – Hilbert spaces and bounded linear operators

3.1 Inner product spaces

⤳ This section is essentially a review of known material (cf. Lindstrøm's book, sect. 5.3).

Set $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We just recall that a Hilbert space H is a vector space (over $\mathbb{F}$) with an inner product such that H is a Banach space w.r.t. to the norm

$$\|x\| := \langle x, x\rangle^{1/2}, \quad x \in H.$$

An important example:

$(X, \mathcal{A}, \mu)$ measure space

Consider $f, g \in \mathcal{L}^2 = \mathcal{L}^2(X, \mathcal{A}, \mu)$. Then $\bar{g}$ is measurable and $\int_X |\bar{g}|^2 \, d\mu = \int_X |g|^2 \, d\mu < \infty$,

So $\bar{g} \in \mathcal{L}^2$. Since 2 is the conjugate exponent of 2 $\left(p = 2 \Rightarrow q = \frac{p}{p-1} = \frac{2}{1} = 2\right)$, we get from Hölder's inequality that $f\bar{g} \in \mathcal{L}^1(X, \mathcal{A}, \mu)$. Hence we may define

$$\boxed{\langle [f], [g] \rangle := \int_X f\bar{g} \, d\mu} \quad \text{for } [f], [g] \in L^2.$$

It is straightforward to check that this gives an inner product on L^2, such that $\|f\|_2 = \langle f, f\rangle^{1/2} = \left(\int_X |f|^2 d\mu\right)^{1/2}$ for all $f \in L^2$.

Since we know that L^2 is a Banach space, we can can conclude that it is a Hilbert space.

In particular, choosing $\mathcal{A} = \mathcal{P}(X)$ and $\mu =$ counting measure we get that $\ell^2(X)$ is a Hilbert space.

For example, if $\{x_n\}, \{y_n\} \in \ell^2 = \ell^2(\mathbb{N})$,

$$\langle \{x_n\}, \{y_n\} \rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}.$$

$$\|x\|_2 = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2}.$$

3.2 Geometry in Hilbert spaces

We let H denote a Hilbert space (over $\mathbb{F}$), $H \neq \{0\}$.

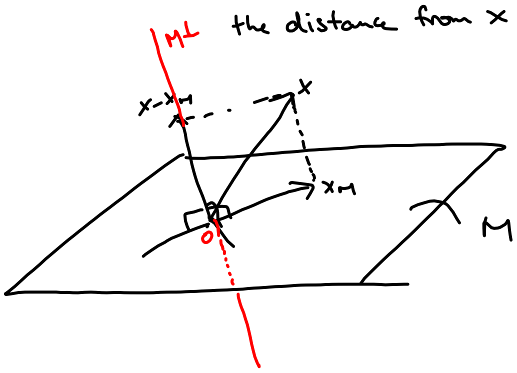
Consider a closed subspace M of H. If M is finite-dimensional, we know how to find the orthogonal projection x_M of $x \in H$ on M (by picking an orth. basis for M); then we know that $x_M \in M$,

$$x - x_M \in M^\perp := \{ z \in X : \langle z, y \rangle = 0 \text{ for all } y \in M \}$$

and $\underbrace{\|x - x_M\|}_{\substack{\text{the distance from} \\ x \text{ to } x_M}} \leq \|x - y\|$ for all $y \in M$

i.e. $\|x - x_M\| = \underbrace{\inf\{\|x-y\|, y \in M\}}_{\text{the distance from } x \text{ to } M}$.

x_M is the best approximation of x in M

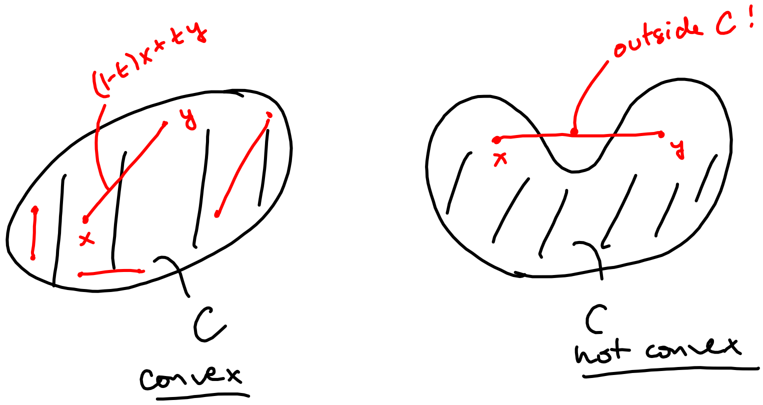


We will show that this also holds when M is infinite-dim.

We first consider a more general situation:

Def. Let C be a non-empty subset of a vector space.

Then C is called convex if $(1-t)x + ty \in C$ for all $x, y \in C$ and all $t \in [0,1]$.

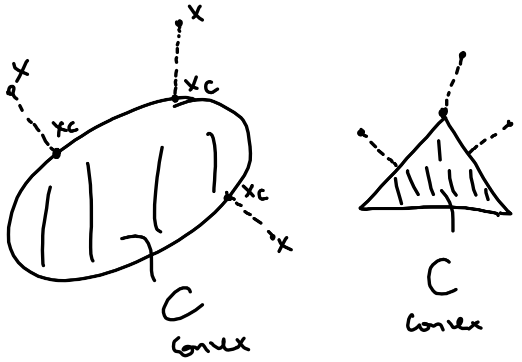


Clearly, a subspace M of any vect. space is convex.

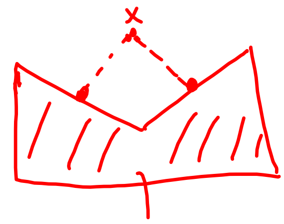
Theorem Assume $C \neq \emptyset$ is a closed, convex subset of a Hilb. space H and let $x \in H$.
Then there is a unique $x_C \in C$ such that

$$\boxed{\|x - x_C\| \leq \|x - y\| \quad \text{for all } y \in C}$$

(x_C is called the best approx. of x in C)



The uniqueness part is important!

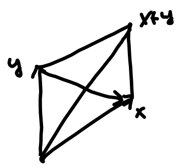


C not convex

x_C is not necessarily unique in this case

Proof of the theorem: [We will use the ◇-law:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$



$$\hookrightarrow = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle = \dots = 2\underbrace{\langle x,x\rangle}_{\|x\|^2} + 2\underbrace{\langle y,y\rangle}_{\|y\|^2}$$]

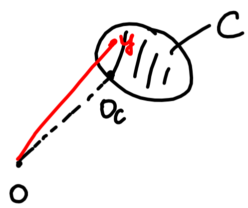
- We first consider the case $x = 0$

We then have to show that there is a unique $0_C \in C$ such that

$$\|0 - 0_C\| \leq \|0 - y\| \quad \text{for all } y \in C$$

i.e. $\|0_C\| \leq \|y\|$ for all $y \in C$.

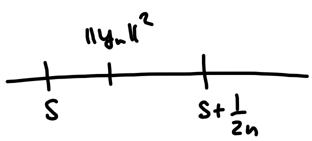
i.e. $\|0_C\| = \inf\{\|y\|, y \in C\}$



Set $s := \inf\{\|y\|^2, y \in C\}$.

For each $n \in \mathbb{N}$, we can find $y_n \in C$ such that

$$(*) \quad s \leq \|y_n\|^2 < s + \frac{1}{2n}$$



We claim that $\{y_n\}$ is a Cauchy-sequence:

Let $m,n \in \mathbb{N}$. Then the $\diamond$-law gives that

$$\|y_n+y_m\|^2 + \|y_n-y_m\|^2 = 2\|y_n\|^2 + 2\|y_m\|^2$$
$$< 2\left(s+\frac{1}{2n}\right) + 2\left(s+\frac{1}{2m}\right)$$
$$= 4s + \frac{1}{n} + \frac{1}{m}$$

Since C is convex, $\frac{1}{2}y_n + \frac{1}{2}y_m \in C$. Hence,

$$\|y_n+y_m\|^2 = 4\,\|\underbrace{\tfrac{1}{2}y_n+\tfrac{1}{2}y_m}_{\in C}\|^2 \geq 4s \quad (\text{by def. of } s)$$

Thus we get that

$$\|y_n-y_m\|^2 < 4s + \frac{1}{n} + \frac{1}{m} - \|y_n+y_m\|^2$$
$$\leq \cancel{4s} + \frac{1}{n} + \frac{1}{m} - \cancel{4s} = \underbrace{\frac{1}{n}+\frac{1}{m}}_{\downarrow 0} \text{ for } n,m \text{ big.}$$

and the claim readily follows.

Since H is complete, $\{y_n\}$ converges to some $y_0 \in H$.

But $\{y_n\} \subseteq C$ and C is closed, so $y_0 \in C$

Letting $n \to \infty$ in $(*)$, we get that $s \leq \underbrace{\lim_{n\to\infty} \|y_n\|^2}_{\|y_0\|^2} \leq s$,

i.e. $\|y_0\| = \sqrt{s} = \inf\{\|y\|, y \in C\}$.

To show that y_0 is the unique vector in C satisfying this, assume $y_0' \in C$ also satisfies $\|y_0'\| = \inf\{\|y\|, y \in C\}$

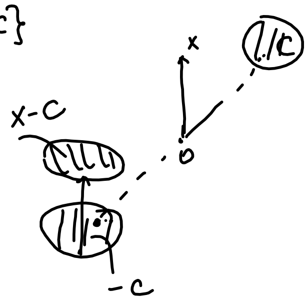
Set $z_n = \begin{cases} y_0, & n = 1,3,5,\dots \\ y_0', & n = 2,4,6,\dots \end{cases}$ Then $\{z_n\} \subseteq C$ and

$$s = \|z_n\|^2 < s + \frac{1}{2n} \quad \text{for all } n \in \mathbb{N}$$

Arguing as above, we get that $\{z_n\}$ is convergent. But this implies that $y_0' = y_0$!!

Thus y_0 is unique, and we can set $0_C := y_0$.

- If $x \neq 0$, set $D := x - C := \{x-y : y \in C\}$



Then D is also closed and convex (check).

So we can find a unique $O_D \in D$ such that

$\|O_D\| = \inf\{\|z\|, z \in D\}$

Since $O_D = x - x_C$ for a unique $x_C \in C$, we get $\|x - x_C\| = \inf\{\|x-y\| : y \in C\}$, as desired.

When C is a closed subspace, we can say more:

Prop. Let M be a closed subspace of H and $x \in H$.
Let x_M be the best appr. of x in M (which exists and is unique by the theorem)
Then $x - x_M \in M^\perp$

Proof: Set $x^\perp := x - x_M$. Have to show that $\langle x^\perp, y\rangle = 0 \ \forall y \in M$.

Let $y \in M$ and $\varepsilon > 0$. Then $x_M + \varepsilon y \in M$, so we get

$$\|x^\perp\|^2 = \|x - x_M\|^2 \leq \|x - (x_M + \varepsilon y)\|^2 = \|x^\perp - \varepsilon y\|^2$$
$$= \langle x^\perp - \varepsilon y, x^\perp - \varepsilon y\rangle$$
$$= \|x^\perp\|^2 \underbrace{- \langle x^\perp, \varepsilon y\rangle - \underbrace{\langle \varepsilon y, x^\perp\rangle}_{= \overline{\langle x^\perp, \varepsilon y\rangle}}}_{-2\,\mathrm{Re}\langle x^\perp, \varepsilon y\rangle} + \|\varepsilon y\|^2$$
$$= \|x^\perp\|^2 - 2\varepsilon\,\mathrm{Re}\langle x^\perp, y\rangle + \varepsilon^2\|y\|^2$$

So $2\cancel{\varepsilon}\,\mathrm{Re}\langle x^\perp, y\rangle \leq \varepsilon^{\cancel{2}}\|y\|^2$ for all $y \in M$

Since $\varepsilon > 0$ is arbitrary, we get that $\mathrm{Re}\langle x^\perp, y\rangle \leq 0$

Since M is a subspace, $-y \in M \ \forall y \in M$. So we also get

$\underbrace{\mathrm{Re}\langle x^\perp, -y\rangle}_{-\mathrm{Re}\langle x^\perp, y\rangle} \leq 0$, i.e. $\mathrm{Re}\langle x^\perp, y\rangle \geq 0$

Hence $\mathrm{Re}\langle x^\perp, y\rangle = 0$. If $\mathbb{F} = \mathbb{R}$, we are done!

If $\mathbb{F} = \mathbb{C}$, then we can use that $iy \in M$ for all $y \in M$.

This gives that $\mathrm{Im}\langle x^\perp, y\rangle = 0$, hence $\langle x^\perp, y\rangle = 0$ $\forall y \in M$

as desired!
