3.2 (ctd) – Orthogonal projections

H Hilbert space, $H \neq \{0\}$.

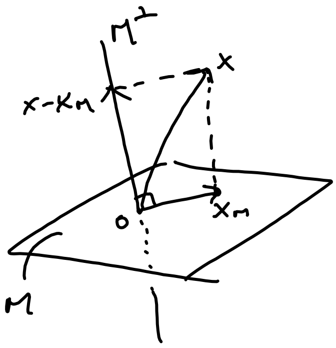
M closed subspace of H, $x \in H$.

Have seen that

- There is a unique $x_M \in M$ such that

$$\|x - x_M\| \leq \|x - y\| \quad \forall y \in M.$$

- x_M satisfies that $x - x_M \in M^\perp$.



We will see soon that it follows that

H is the "direct sum" of M and $M^\perp$.

Def. Let M_1, M_2 be subspaces of H.

- H is called the algebraic direct sum of M_1 and M_2 when every $x \in H$ can be written in a unique way as

$$x = m_1 + m_2, \quad \text{where } m_1 \in M_1 \text{ and } m_2 \in M_2.$$

- If this happens and M_1, M_2 are both closed, then we say that H is the direct sum of M_1 and M_2, and write $\boxed{H = M_1 \oplus M_2}$

Note: H is the algebraic sum of M_1 and M_2

$\Leftrightarrow H = \{m_1 + m_2 : m_1 \in M_1, m_2 \in M_2\}$ and $M_1 \cap M_2 = \{0\}$.

[This is easy – see the notes if needed!]

Theorem Let M be a closed subspace of H.

- Then $M^\perp$ is a closed subspace of H and $\boxed{H = M \oplus M^\perp}$
- The map $P_M : H \to H$ given by $\boxed{P_M(x) = x_M}$ for $x \in H$ is called the orthogonal projection of H on M. It enjoys the foll. properties:
  - P_M is linear and bounded, with $\|P_M\| = 1$ (if $M \neq \{0\}$)
  - $(P_M)^2 = P_M$
  - $P_M(H) = M$, $\ker(P_M) = M^\perp$
- Moreover, we have that $\boxed{(M^\perp)^\perp = M}$, and $\boxed{P_{M^\perp} = I_H - P_M}$

I_H: the identity operator on H.

Proof . It is easy to check that $M^\perp$ is a closed subspace (Exercise).

- For $x \in H$, we have $x = \underbrace{x_M}_{\in M} + \underbrace{(x - x_M)}_{\in M^\perp}$. So H is the alg. direct sum of M and $M^\perp$. As $M \cap M^\perp = \{0\}$ (because if $x \in M \cap M^\perp$, then $\langle \underbrace{x}_{\in M}, \underbrace{x}_{\in M^\perp} \rangle = 0$, i.e. $x = 0$), we get that $H = M \oplus M^\perp$

- Linearity of P_M is left as an exercise.
- Boundedness: For $x \in H$, we have

  $$\|P_M(x)\|^2 = \|x_M\|^2 \leq \|x_M\|^2 + \|x - x_M\|^2 \underset{\uparrow \text{ Pythagoras}}{=} \|x\|^2$$

  So P_M is bounded, with $\|P_M\| \leq 1$.

  If $M \neq \{0\}$, and $y \in M$, $\|y\| = 1$, then $\|P_M(y)\| = \|y\| = 1$, so $\|P_M\| \geq 1$, and we get that $\|P_M\| = 1$.

- $(P_M)^2 = P_M$: $(P_M)^2(x) = P_M(\underbrace{P_M(x)}_{x_M \in M}) = x_M = P_M(x)$

- $P_M(H) = M$ is obvious.
- $\mathrm{Ker}(P_M) = M^\perp$: If $x \in \mathrm{Ker}(P_M)$, i.e. $x_M = 0$, then $x = x - x_M \in M^\perp$. Conversely, let $x \in M^\perp$. Then

  $$x_M = \underbrace{x}_{\in M^\perp} + \underbrace{(x_M - x)}_{\in M^\perp} \in M^\perp,$$

  So $x_M \in M \cap M^\perp = \{0\}$, i.e. $x \in \mathrm{Ker}(P_M)$

- $(M^\perp)^\perp = M$:

  $M \subseteq (M^\perp)^\perp$: Let $y \in M$. For any $z \in M^\perp$, we have $\langle y, z \rangle = 0$. So $y \in (M^\perp)^\perp$

  $(M^\perp)^\perp \subseteq M$: We have $H = M^\perp \oplus (M^\perp)^\perp$

  Let $x \in (M^\perp)^\perp$, and set $x^\perp := x - x_M \in M^\perp$.

  Since $x_M \in M \subseteq (M^\perp)^\perp$, we have

  $$x = \underbrace{0}_{\in M^\perp} + \underbrace{x}_{\in (M^\perp)^\perp} \quad \text{and} \quad x = \underbrace{x^\perp}_{\in M^\perp} + \underbrace{x_M}_{\in (M^\perp)^\perp}$$

  By uniqueness of dec. in a direct sum, we get that ($0 = x^\perp$ and) $x = x_M \in M$.

- $P_{M^\perp} = I_H - P_M$: We have $x = \underbrace{(x - x_M)}_{\in M^\perp} + \underbrace{x_M}_{\in M = (M^\perp)^\perp}$ and $x = \underbrace{x_{M^\perp}}_{\in M^\perp} + \underbrace{(x - x_{M^\perp})}_{\in (M^\perp)^\perp}$

  So we get by uniqueness that $x_{M^\perp} = x - x_M$

  i.e. $P_{M^\perp}(x) = x - x_M = (I_H - P_M)(x)$

Example $H = L^2(X, \mathcal{A}, \mu)$, $E \in \mathcal{A}$.

Define $M_E = \{ [g] : g \in \mathcal{L}^2, \underbrace{\mu(\{x \in E^c \mid g(x) \neq 0\}) = 0}_{\text{"g lives essentially on E"}} \}$.

Then, setting $F = E^c \in \mathcal{A}$, it is not difficult to verify that $(M_E)^\perp = M_F$, and similarly $(M_F)^\perp = M_E$. (see the notes.)

So M_E and M_F are closed subspaces and we have

$$H = M_E \oplus (M_E)^\perp = M_E \oplus M_F$$

Since $[f] = \underbrace{[f 1_E]}_{\in M_E} + \underbrace{[f 1_F]}_{\in M_F}$, we get that $P_{M_E}([f]) = [f 1_E]$

and $P_{M_F}([f]) = [f 1_F]$.

Corollary $\boxed{\text{Let } M \text{ be a closed subspace of } H. \text{ Then } M = H \iff M^\perp = \{0\}.}$

(($\Rightarrow$) is trivial. ($\Leftarrow$): If $M^\perp = \{0\}$, then $M = (M^\perp)^\perp = \{0\}^\perp = H$)

Corollary

Let $\emptyset \neq S \subseteq H$. Then

$$\underbrace{\overline{\mathrm{Span}(S)}^{\|\cdot\|} = H}_{} \iff S^\perp = \{0\}$$

i.e. Span(S) is dense in H

Here, $S^\perp := \{ z \in H : \langle z, s \rangle = 0 \text{ for all } s \in S \}$.

Proof Set $M := \overline{\mathrm{Span}(S)}^{\|\cdot\|}$, so M is a closed subsp. of H.

Then $M = H \iff M^\perp = \{0\}$ by the prev. corollary.

But $M^\perp = S^\perp$ (exercise), so we are done.

3.3 Orthonormal bases

H Hilb. space, $H \neq \{0\}$.

Def. Let $\mathcal{B}$ be a nonempty subset of H. We say that $\mathcal{B}$ is an orthonormal basis (o.n.b.) for H when $\mathcal{B}$ is orthonormal and $\mathrm{Span}(\mathcal{B})$ is dense in H.

(If H is finite-dimensional, this gives the usual concept.)

Proposition If $\mathcal{B}$ is orthormal, then $\mathcal{B}$ is an o.n.b. for H $\Leftrightarrow$ $\mathcal{B}^\perp = \{0\}$.

This follows immediately from the last corollary in 3.2.

Example $\emptyset \neq X$, $H = \ell^2(X)$ (so $\mathcal{A} = \mathcal{P}(X)$, μ = counting measure)

For $x \in X$, let $e_x \in H$ be given by $e_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$

Then $\mathcal{B} := \{e_x\}_{x \in X}$ is an o.n.b. for H (i.e., $e_x = \delta_{\{x\}}$)

Indeed, we have $\langle e_x, e_{x'} \rangle = \int_X e_x \overline{e_{x'}}\, d\mu = \int_X 1_{\{x\}} 1_{\{x'\}}\, d\mu$

$$= \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{if } x \neq x' \end{cases}$$

So $\mathcal{B}$ is orthonormal.

> Note: one often writes $\sum_{x \in X} f(x) = \int_X f\, d\mu$ when $f \in \ell^1(X)$

Moreover, if $f \in \mathcal{B}^\perp$, then for each $x \in X$ we have

$$0 = \langle f, e_x \rangle = \int_X f\, \overline{e_x}\, d\mu = \int_X f\, 1_{\{x\}}\, d\mu = \int_{\{x\}} f\, d\mu = f(x),$$

So $f = 0$. Hence, $\mathcal{B}^\perp = \{0\}$. So $\mathcal{B}$ is an o.n.b. for H.

Example Assume $\{x_j\}_{j\in\mathbb{N}}$ is a sequence in a Hilbert space H such that $\operatorname{Span}\{x_j : j\in\mathbb{N}\}$ is dense in H.

Then we can apply the Gram-Schmidt process to produce ~~as~~ a countable o.n.b. for H:

- $y_1 := x_1$; $\mathcal{B}_1 := \{ \frac{1}{\|y_1\|} y_1 \}$.
- Let $n\geq 1$ and suppose we have constructed an o.n.b. $\mathcal{B}_n$ for $M_n := \operatorname{span}\{x_1, \dots, x_n\}$.

  If $x_{n+1} \in M_n$, then set $\mathcal{B}_{n+1} := \mathcal{B}_n$.

  Otherwise, set $y_{n+1} := x_{n+1} - P_{M_n}(x_{n+1}) \in (M_n)^\perp$

  and set $\mathcal{B}_{n+1} := \mathcal{B}_n \cup \{ \frac{1}{\|y_{n+1}\|} y_{n+1} \}$.

Then $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ is an o.n.b. for H (which is countable).