3.3 Orthonormal bases II

H Hilbert space $\neq \{0\}$.

Let $\mathcal{B}$ be an orthonormal subset of H. We have seen that

$\mathcal{B}$ is an o.n.b. for H $\underset{\text{(by def.)}}{\iff}$ Span$(\mathcal{B})$ is dense in H

$\iff$ $\mathcal{B}^\perp = \{0\}$.

An important example from Fourier analysis:

Let $H = L^2([-\pi,\pi], \mathcal{A}, \mu)$ where $\mathcal{A}$ = Leb. meas. subsets of $[-\pi,\pi]$ and μ is the normalized Leb. measure on $\mathcal{A}$, i.e. $\mu = \frac{1}{2\pi}\lambda$ with λ the usual Leb. measure so that $\mu([-\pi,\pi]) = \frac{1}{2\pi}2\pi = \underline{\underline{1}}$.

For each $n \in \mathbb{Z}$, define $e_n : [-\pi,\pi] \to \mathbb{C}$ by $\boxed{e_n(t) = e^{int}, \quad |t| \leq \pi.}$

Then $\mathcal{B} = \{[e_n], n \in \mathbb{Z}\}$ is orthonormal.

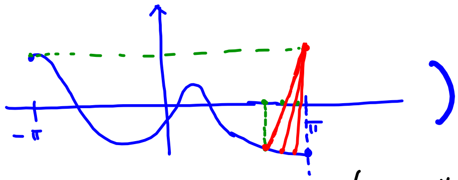
Moreover, $\mathcal{B}$ is an o.n.b. for H:

Let $[f] \in H$, $\varepsilon > 0$. Using a result from chap. 2 in ELA, we can find $g \in C([-\pi,\pi])$ s.t. $\| [f] - [g] \|_2 < \varepsilon/3$.

Next, we can find a periodic $k \in C([-\pi,\pi])$ such that $\| [g] - [k] \|_2 < \varepsilon/3$

[periodic means that $k(-\pi) = k(\pi)$]

(Exercise. The idea is as follows.



)

Finally, as shown in Lindstrøm's book (Cor. 4.11.13), we can find $h \in \mathrm{Span}\{e_n, n \in \mathbb{Z}\}$ s.t. $\|k-h\|_u < \varepsilon/3$ ← the uniform norm

Since

$$\| [k] - [h] \|_2^2 = \int_{[-\pi,\pi]} |k-h|^2 \, d\mu \leq \|k-h\|_u^2 \underbrace{\int_{[-\pi,\pi]} 1 \, d\mu}_{\mu([-\pi,\pi]) = 1},$$

we have $\| [k] - [h] \|_2 < \varepsilon/3$,

so we get

$$\| [f] - [h] \|_2 \underset{\Delta\text{-ineq.}}{\leq} \underbrace{\| [f] - [g] \|_2}_{< \varepsilon/3} + \underbrace{\| [g] - [k] \|_2}_{< \varepsilon/3} + \underbrace{\| [k] - [h] \|_2}_{< \varepsilon/3} < \varepsilon,$$

which shows that span$(\mathcal{B})$ is dense in H (w.r.t. $\|\cdot\|_2$), as desired.

Note It can shown (in a non-constructive way) that any Hilbert space $\neq \{0\}$ has an o.n.b.

What are o.n.b. good for ?!

Lemma Assume $\mathcal{B} \subseteq H$ is orthonormal and $x \in H$.

Then $\sum_{u \in \mathcal{B}} |\langle x, u\rangle|^2 \leq \|x\|^2$ Bessel's inequality (general case)

Hence, $\mathcal{B}_x := \{u \in \mathcal{B} : \langle x, u\rangle \neq 0\}$ is countable (cf. Extra-Exercise 1).

Proof Let $\emptyset \neq F \subseteq \mathcal{B}$, F finite, say $F = \{u_1, \dots, u_n\}$ (without repetition)

Then F is orthonormal, so the "usual" Bessel's inequality gives that $\sum_{u \in F} |\langle x, u\rangle|^2 = \sum_{j=1}^{n} |\langle x, u_j\rangle|^2 \leq \|x\|^2$

(the point here is to use that the orth. proj. of x on $\mathrm{span}(F)$ is given $\sum_{j=1}^{n} \langle x, u_j\rangle u_j$)

Taking the sup. over all such F's give the result.

The next lemma is Prop. 5.3.11 in Lindstrøm's book:

Lemma Assume $\{u_j : j \in \mathbb{N}\}$ is an orthonormal subset of H, and $\{c_j\}_{j \in \mathbb{N}}$ is a sequence in $\mathbb{C}$ s.t. $\sum_{j=1}^{\infty} |c_j|^2 < \infty$ (i.e. $\{c_j\}_{j \in \mathbb{N}} \in \ell^2(\mathbb{N})$).

Then $\sum_{j=1}^{\infty} c_j u_j$ converges to some $y \in H$, and we have $c_j = \langle y, u_j\rangle$ for all $j \in \mathbb{N}$.

(The idea is to check that $\left\{\sum_{j=1}^{n} c_j u_j\right\}_{n \in \mathbb{N}}$ is a Cauchy-seq. in H).

Theorem Let $\mathcal{B} \subseteq H$ be orthonormal. The the foll. conditions are equivalent:

(a) $\mathcal{B}$ is an o.n.b. for H

(b) Every $x \in H \setminus \{0\}$ has a "Fourier expansion" w.r.t. $\mathcal{B}$

$$(*) \quad \boxed{x = \sum_{u \in \mathcal{B}_x} \langle x, u\rangle u}$$

where $\mathcal{B}_x = \{u \in \mathcal{B} : \langle x, u\rangle \neq 0\}$ is countable.

(Here, if $\mathcal{B}_x$ is infinite, say $\mathcal{B}_x = \{u_j : j \in \mathbb{N}\}$ (without rep.), then $(*)$ means that $x = \sum_{j=1}^{\infty} \langle x, u_j\rangle u_j$.)

(c) For every $x \in H$, we have $\boxed{\|x\|^2 = \sum_{u \in \mathcal{B}} |\langle x, u\rangle|^2}$.

Parseval's identity

Proof (a) $\Rightarrow$ (b): Let $\mathcal{B}$ be an o.n.b. for H. So we know that $\mathcal{B}^\perp = \{0\}$.

Let $x \in H \setminus \{0\}$. Note that $\mathcal{B}_x \neq \emptyset$ (otherwise we would have $x \in \mathcal{B}^\perp = \{0\}$)

We assume that $\mathcal{B}_x$ is infinite (which is more difficult).

Say $\mathcal{B}_x = \{u_j : j \in \mathbb{N}\}$ (without rep.). Bessel's inequality gives that $\sum_{j=1}^{\infty} |\langle x, u_j\rangle|^2 \leq \|x\|^2$, so

$$\{\underbrace{\langle x, u_j\rangle}_{c_j}\}_{j\in\mathbb{N}} \in \ell^2(\mathbb{N})$$

So the previous lemma gives that $\sum_{j=1}^{\infty} \underbrace{\langle x, u_j\rangle}_{c_j} u_j = y$ for some $y \in H$,

and $\langle x, u_j\rangle = \langle y, u_j\rangle$ for all $j \in \mathbb{N}$.

Now, if $u \in \mathcal{B} \setminus \mathcal{B}_x$, then $\langle x, u\rangle = 0$, but we also have

$$\langle y, u\rangle = \left\langle \sum_{j=1}^{\infty} c_j u_j, u\right\rangle \underset{\text{of the inner product in the 1st var.}}{\overset{\text{cont.}}{=}} \sum_{j=1}^{\infty} c_j \underbrace{\langle u_j, u\rangle}_{\overset{\shortparallel}{0} \text{ for all } j} = 0$$

This means that $\langle x, u\rangle = \langle y, u\rangle$ for all $u \in \mathcal{B}$,

i.e. $\langle x-y, u\rangle = 0$

i.e. $x - y \in \mathcal{B}^\perp$

But $\mathcal{B}^\perp = \{0\}$, so we get $x = y$, so $x = \sum_{j=1}^{\infty} \langle x, u_j\rangle u_j$ as desired.

(b) $\Rightarrow$ (c): Assume (b) holds and $x \in H \setminus \{0\}$.

Again we consider only the case where $\mathcal{B}_x = \{u_j, j \in \mathbb{N}\}$.

We then have $x = \sum_{j=1}^{\infty} \langle x, u_j\rangle u_j$, so

$$\|x\|^2 \underset{\substack{\uparrow \\ \text{the norm} \\ \text{is cont.}}}{=} \lim_{n\to\infty} \left\| \sum_{j=1}^{n} \langle x, u_j\rangle u_j \right\|^2 \underset{\substack{\uparrow \\ \text{Pythagoras}}}{=} \lim_{n\to\infty} \sum_{j=1}^{n} \underbrace{\|\langle x, u_j\rangle u_j\|^2}_{= |\langle x, u_j\rangle|^2}$$

$$= \sum_{j=1}^{\infty} |\langle x, u_j\rangle|^2 .$$

So $\sum_{u\in\mathcal{B}} |\langle x, u\rangle|^2 = \sum_{u\in\mathcal{B}_x} |\langle x, u\rangle|^2 = \sum_{j=1}^{\infty} |\langle x, u_j\rangle|^2 = \|x\|^2$.

(c) $\Rightarrow$ (a) Assume $\|x\|^2 = \sum_{u\in\mathcal{B}} |\langle x, u\rangle|^2$ for all $x \in H$.

If $x \in \mathcal{B}^\perp$, i.e. $\langle x, u\rangle = 0$ for all $u \in \mathcal{B}$,

then we get that $\|x\|^2 = 0$, i.e. $x = 0$.

So $\mathcal{B}^\perp = \{0\}$, and it follows that $\mathcal{B}$ is an o.n.b. for H.

Corollary Assume $\mathcal{B}$ is a countable orthonormal subset of H, say $\mathcal{B} = \{v_k, k \in \mathbb{N}\}$ (without rep.). Then

$$\boxed{\begin{aligned} \mathcal{B} \text{ is an o.n.b. for } H &\iff x = \sum_{k=1}^{\infty} \langle x, v_k\rangle v_k \text{ for every } x \in H \\ &\iff \|x\|^2 = \sum_{k=1}^{\infty} |\langle x, v_k\rangle|^2 \quad \text{— ॥ —} \end{aligned}}$$

Example (Fourier coefficients)

$H = L^2([-\pi,\pi])$ as before, with normalized Leb. measure μ,

so $\mathcal{B} = \{[e_n], n \in \mathbb{Z}\}$ is an o.n.b for H.

Let $[f] \in H$, $n \in \mathbb{Z}$. Then

$$\langle [f], [e_n] \rangle = \int_{[-\pi,\pi]} f\, \overline{e_n}\, d\mu = \int_{[-\pi,\pi]} f(t)\, e^{-int}\, d\mu(t)$$

$$= \frac{1}{2\pi} \int_{[-\pi,\pi]} f(t)\, e^{-int}\, d\lambda(t)$$

usual Leb. meas.

which is called the Fourier coefficient of $[f]$ at n

and usually denoted by $\widehat{[f]}(n)$, or

in fact just $\hat{f}(n)$ if we identify implicitely functions agreeing almost everywhere. We then get from the theorem that

$$\boxed{f = \lim_{m\to\infty} \sum_{n=-m}^{m} \hat{f}(n)\, e_n \quad (\text{w.r.t. } \|\cdot\|_2)}$$

i.e. "$f = \sum_{n=-\infty}^{\infty} \hat{f}(n)\, e_n$"

and $$\boxed{\|f\|_2^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2}$$