Solutions to the exam in MAT3400/4400, Spring 2024

Problem 1

(a) Formulate what it means that a measure is absolutely continuous with respect to another measure. Formulate the Radon–Nikodym derivative theorem.

(b) Consider the set $\mathbb{N}$ of natural numbers. Take two sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ of strictly positive numbers and consider two measures μ and ν on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ (where $\mathcal{P}(\mathbb{N})$ denotes the σ-algebra of all subsets of $\mathbb{N}$) defined by

$$\mu(A) = \sum_{n\in A} a_n, \qquad \nu(A) = \sum_{n\in A} b_n.$$

What is the Radon–Nikodym derivative $\dfrac{d\mu}{d\nu}$? Justify the answer.

Solution:

(a) Given a measurable space $(X, \mathcal{B})$ and two measures μ and ν on it, we say that μ is absolutely continuous with respect to ν, if for every $A \in \mathcal{B}$ such that $\nu(A) = 0$ we have $\mu(A) = 0$.

The Radon–Nikodym theorem has several slightly different formulations, any of which is accepted as a correct answer. The most common form states that if μ is absolutely continuous with respect to ν and both measures are σ-finite, then there is a measurable function $f\colon X \to [0, +\infty)$ such that $d\mu = f\,d\nu$, that is,

$$\mu(A) = \int_A f\,d\nu \quad \text{for all} \quad A \in \mathcal{B}.$$

Moreover, f is essentially unique in the sense that if $\tilde{f}$ is another function with the same properties, then $f = \tilde{f}$ ν-a.e.

(b) Since $a_n > 0$ and $b_n > 0$ for all n, the only set that has measure zero with respect to either measure is the empty set. In particular, μ is absolutely continuous with respect to ν, hence a Radon–Nikodym derivative $f = \dfrac{d\mu}{d\nu}$ exists. For every $n \in \mathbb{N}$ we have

$$a_n = \mu(\{n\}) = \int_{\{n\}} f\,d\nu = f(n)\nu(\{n\}) = f(n)b_n, \quad \text{hence} \quad f(n) = \frac{a_n}{b_n}.$$

Problem 2

Determine whether the following sequences $(f_n)_{n=1}^{\infty}$ converge in $L^1(0,1)$ (we consider the Lebesgue measure on $(0,1)$), and whenever they do, find the limits. Justify your answer.

(a) $f_n(x) = n\mathbb{1}_{(0,n^{-1}]}(x)$, where $\mathbb{1}_A$ denotes the characteristic function of a set A.

(b) $f_n(x) = \dfrac{\sin x^n}{x^n}$.

Solution:

(a) The sequence $(f_n)_n$ does not converge in $L^1(0,1)$. One way to see this is to check that it is not a Cauchy sequence: if $n > m$, then

$$\|f_n - f_m\|_1 = \int_{(0,1)} |f_n - f_m|\,d\lambda = \int_{(0,1/n)} (n-m)\,d\lambda + \int_{(1/n,1/m)} m\,d\lambda = 2 - \frac{2m}{n},$$

and the last expression does not converge to zero when n and m (with $n \geq m$) go to infinity.

Another possibility is to argue as follows. If the sequence $(f_n)_n$ converges in $L^1(0,1)$ to a function g, then a subsequence converges to g a.e. But we have $f_n \to 0$ pointwise, so $g = 0$. Therefore $f_n \to 0$ in $L^1(0,1)$, which is nonsense as $\|f_n\|_1 = 1$ for all n.

(b) We use as known facts that $0 \le \sin x \le x$ for $x \in (0,1)$ and $\dfrac{\sin x}{x} \to 1$ as $x \downarrow 0$.

As $x^n \to 0$ for every $x \in (0,1)$, we thus see that $f_n(x) \to 1$ for all $x \in (0,1)$. As $|f_n(x)| \le 1$ for all $n \in \mathbb{N}$ and $x \in (0,1)$, by the dominated convergence theorem it follows that $f_n \to 1$ (the constant function 1) in $L^1(0,1)$.

We may also recall/observe that the function $\dfrac{\sin x}{x}$ is monotonically decreasing on $(0,1)$. Therefore the sequence $(f_n)_n$ is monotonically increasing, so instead of the dominated convergence theorem we could have used the monotone convergence theorem.

Problem 3

(a) Formulate the Tonelli theorem.

(b) Assume $(X, \mathcal{B}, \mu)$ is a σ-finite measure space and $f\colon X \to [0,+\infty)$ is a measurable function. For every $t \ge 0$ consider the set $A_t = \{x \in X : f(x) \ge t\}$. Show that

$$\int_X f\,d\mu = \int_{[0,+\infty)} \mu(A_t)\,d\lambda(t),$$

where λ denotes the Lebesgue measure. Hint: integrate the characteristic function of the set $A = \{(t,x) : f(x) \ge t\}$.

(c) In the setting of (b), show that for every number $p > 0$ we have

$$\int_X f^p\,d\mu = p\int_{[0,+\infty)} t^{p-1}\mu(A_t)\,d\lambda(t).$$

Solution:

(a) A short accepted formulation of the Tonelli theorem is that if $(X_1, \mathcal{B}_1, \mu_1)$ and $(X_2, \mathcal{B}_2, \mu_2)$ are σ-finite measure spaces, $f\colon X_1 \times X_2 \to [0,+\infty]$ is a $(\mathcal{B}_1 \times \mathcal{B}_2)$-measurable function, then the integrals

$$\int_{X_1\times X_2} f\,d(\mu_1\times\mu_2),\quad \int_{X_1}\left(\int_{X_2} f(x_1,x_2)d\mu_2(x_2)\right)d\mu_1(x_1),\quad \int_{X_2}\left(\int_{X_1} f(x_1,x_2)d\mu_1(x_1)\right)d\mu_2(x_2)$$

are well-defined (but possibly infinite) and equal.

(b) Following the hint we integrate $\mathbb{1}_A$ with respect to the measure $\lambda \times \mu$ on $[0,+\infty) \times X$. By the Tonneli theorem we get

$$(\lambda\times\mu)(A) = \int_X\left(\int_{[0,+\infty)} \mathbb{1}_A(t,x)\,d\lambda(t)\right)d\mu(x) = \int_X\left(\int_{\{t:0\le t\le f(x)\}} 1\,d\lambda(t)\right)d\mu(x) = \int_X f(x)\,d\mu(x),$$

$$(\lambda\times\mu)(A) = \int_{[0,+\infty)}\left(\int_X \mathbb{1}_A(t,x)\,d\mu(x)\right)d\lambda(t) = \int_{[0,+\infty)} \mu(A_t)\,d\lambda(t),$$

so the two integrals in the formulation of the problem are indeed equal.

(c) One possibility is to argue similarly to (b), but this time integrate the function $(t,x) \mapsto pt^{p-1}\mathbb{1}_A(t,x)$. Another possibility is to use (b) together with the change of variables formula for the Lebesgue measure. Namely, by (b) applied to f^p we have

$$\int_X f^p\,d\mu = \int_{[0,+\infty)} \mu(\{x : f(x)^p \ge t\})\,d\lambda(t) = \int_{[0,+\infty)} \mu(\{x : f(x) \ge t^{1/p}\})\,d\lambda(t).$$

Make the change of variables $t = s^p$. Then the last integral becomes equal to

$$\int_{[0,+\infty)} \mu(\{x : f(x) \ge s\})ps^{p-1}\,d\lambda(s) = p\int_{[0,+\infty)} s^{p-1}\mu(A_s)\,d\lambda(s),$$

which is what we need. (To be precise, we apply the change of variables formula to the integral over $(0, +\infty)$. Point 0 does not play any role, as its Lebesgue measure is zero.)

Problem 4

Consider the space $\mathcal{S}(\mathbb{R})$ of Schwartz functions on $\mathbb{R}$. Show using the Fourier transform that for every $g \in \mathcal{S}(\mathbb{R})$ there exists a unique function $f \in \mathcal{S}(\mathbb{R})$ such that

$$f'' - f = g.$$

Solution:

Recall that the Fourier transform $h \mapsto \widehat{h}$ is a bijection of $\mathcal{S}(\mathbb{R})$ onto itself. Hence the equation $f'' - f = g$ is equivalent to $\widehat{f''} - \widehat{f} = \widehat{g}$. If we use the same conventions for the Fourier transform as in the lectures, that is,

$$\widehat{h}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} h(y) e^{-ixy} dy,$$

then $\widehat{h'}(x) = ix\,\widehat{h}(x)$, hence $\widehat{h''}(x) = -x^2 \widehat{h}(x)$. Therefore the equation $f'' - f = g$ is equivalent to

$$-(x^2 + 1)\widehat{f}(x) = \widehat{g}(x).$$

This equation has a unique solution $\widehat{f}(x) = -\dfrac{\widehat{g}(x)}{1 + x^2}$. Note that the last fraction indeed defines a function in $\mathcal{S}(\mathbb{R})$.

Problem 5

Assume $f\colon \mathbb{R} \to [0, +\infty)$ is a Lebesgue measurable integrable (with respect to the Lebesgue measure) function. For $x \in \mathbb{R}$, define

$$g(x) = \liminf_{n \to +\infty} f(x + n).$$

(a) Show that the function g is integrable.

(b) Show that g is periodic and conclude that $g = 0$ a.e.

Solution:

(a) For every $n \in \mathbb{N}$, consider the function f_n defined by $f_n(x) = f(x + n)$. Then $g(x) = \liminf_{n \to +\infty} f_n(x)$. Hence g is a positive measurable function and by Fatou's lemma we have

$$\int_{\mathbb{R}} g\, d\lambda \leq \liminf_{n \to +\infty} \int_{\mathbb{R}} f_n\, d\lambda = \int_{\mathbb{R}} f\, d\lambda < \infty,$$

since $\int_{\mathbb{R}} f_n\, d\lambda = \int_{\mathbb{R}} f\, d\lambda$ for all n.

(b) The sequences $(f(x + n))_{n=1}^{\infty}$ and $(f(x + n + 1))_{n=1}^{\infty}$ has the same limit inferior. Hence $g(x) = g(x + 1)$, so g is 1-periodic. Then

$$\int_{\mathbb{R}} g\, d\lambda = \sum_{n \in \mathbb{Z}} \int_{[n,n+1)} g\, d\lambda = \sum_{n \in \mathbb{Z}} \int_{[0,1)} g\, d\lambda,$$

where the first equality follows, for example, from the monotone convergence theorem. As $\int_{\mathbb{R}} g\, d\lambda < \infty$ by (a), the above equality is possible only if both $\int_{[0,1)} g\, d\lambda$ and $\int_{\mathbb{R}} g\, d\lambda$ are zero. As $g \geq 0$, it follows then that $g = 0$ a.e.