Mandatory assignment in MAT4410 Fall 2017

The solution to the assignment must be submitted no later than Friday, 3rd of November at 18:30 by using the electronic system Devilry (https://devilry.ifi.uio.no). You are expected to familiarise yourself with the rules for mandatory assignments available at uio.no/english/studies/examinations/compulsory-activities/mn-math-mandatory.html.

Note: You must provide details for all your answers. The individual questions have different weights, as indicated. To pass the assignment you will need a score of at least 60 points.

Problem 1. (10 points) Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space. Suppose that ν_1, ν_2 are σ-finite measures on $\mathcal{A}$ such that $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$. Form the measure $\nu := \nu_1 + \nu_2$ on $\mathcal{A}$. Prove that $\nu \ll \mu$ and that

$$\frac{d\nu}{d\mu} = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu} \quad \mu\text{-a.e.}$$

Problem 2. **(a)** (5 points) Let Ω be a nonempty set and $\mathcal{C}$ a countable set of pairwise disjoint subsets of Ω such that $\Omega = \bigcup_{E\in\mathcal{C}} E$. Form the σ-algebra $\sigma(\mathcal{C})$ generated by $\mathcal{C}$. Suppose that $f : \Omega \to \mathbb{R}$ is $\sigma(\mathcal{C})$-measurable. Prove that

$$f = \sum_{E\in\mathcal{C}} a_E \chi_E$$

for some real numbers a_E, $E \in \mathcal{C}$.

(b) (5 points) Let $(\Omega, \mathcal{A}, P)$ be a probability space and suppose $F_1, \dots, F_n$ are pairwise disjoint sets in $\mathcal{A}$ such that

$$\Omega = \bigcup_{j=1}^{n} F_j$$

and $P(F_j) > 0$ for all $j = 1, \dots, n$. Let $\mathcal{G} = \sigma(\{F_1, \dots, F_n\})$ and let $f \in \mathcal{L}^1(\Omega, \mathcal{A}, P)$. Prove that $\mathcal{E}(f \mid \mathcal{G})$, the conditional expectation of f given $\mathcal{G}$, is of the form

$$\mathcal{E}(f \mid \mathcal{G}) = \sum_{j=1}^{n} a_j \chi_{F_j}$$

for suitable $a_j \in \mathbb{C}$, $j = 1, \dots, n$.

Problem 3. Let $(\Omega, \mathcal{A}, P)$ be a probability space and suppose $\mathcal{G}$ is a σ-algebra of subsets of Ω with $\mathcal{G} \subseteq \mathcal{A}$. It is assumed known that for each $f \in \mathrm{L}^1(\Omega, \mathcal{A}, P)$ there is a function $\mathcal{E}(f \mid \mathcal{G})$ in $\mathrm{L}^1(\Omega, \mathcal{G}, P|_{\mathcal{G}})$, called the conditional expectation of f given $\mathcal{G}$. Prove that the map $\mathcal{E}(\cdot \mid \mathcal{G}) : \mathrm{L}^1(\Omega, \mathcal{A}, P) \to \mathrm{L}^1(\Omega, \mathcal{G}, P|_{\mathcal{G}})$ satisfies the following properties:

(a) (10 points) $\mathcal{E}(af + bg \mid \mathcal{G}) = a\mathcal{E}(f \mid \mathcal{G}) + b\mathcal{E}(g \mid \mathcal{G})$ for all $f, g \in \mathrm{L}^1(\Omega, \mathcal{A}, P)$ and $a, b \in \mathbb{C}$ (linearity of $\mathcal{E}(\cdot \mid \mathcal{G}) : \mathrm{L}^1(\Omega, \mathcal{A}, P) \to \mathrm{L}^1(\Omega, \mathcal{G}, P|_{\mathcal{G}})$.)

(b) (10 points) $\mathcal{E}(f \mid \mathcal{G}) \geq 0$ P-a.e. for any $f \in \mathrm{L}^1(\Omega, \mathcal{A}, P)$ nonnegative.

(c) (10 points) $\mathcal{E}(fg \mid \mathcal{G}) = g\mathcal{E}(f \mid \mathcal{G})$ P-a.e. for any $f \in \mathrm{L}^1(\Omega, \mathcal{A}, P)$ and $g \in \mathrm{L}^\infty(\Omega, \mathcal{G}, P|_\mathcal{G})$.

In the remaining part of the assignment it is assumed known that the spaces of complex-valued sequences

$$\begin{aligned} \ell^\infty(\mathbb{N}) &= \{x = \{x_n\}_{n\geq 1} \mid \sup_n |x_n| < \infty\} \text{ with } \|x\|_\infty = \sup_n |x_n|, \\ c_0(\mathbb{N}) &= \{x = \{x_n\}_{n\geq 1} \in \ell^\infty \mid \lim_{n\to\infty} x_n = 0\}, \\ \ell^1(\mathbb{N}) &= \{x = \{x_n\}_{n\geq 1} \mid \sum_{n=1}^\infty |x_n| < \infty\} \text{ with } \|x\|_1 = \sum_{n=1}^\infty |x_n| \end{aligned}$$

are Banach spaces for their respective norms.

Problem 4.

(a) (20 points) Prove that $\ell^1(\mathbb{N})$ is the dual of $c_0(\mathbb{N})$, more precisely prove the following three claims: for each $x = \{x_j\}_{j\geq 1}$ in $\ell^1(\mathbb{N})$, the map $\varphi_x : c_0(\mathbb{N}) \to \mathbb{C}$ given by

$$\varphi_x(y) = \sum_{j=1}^\infty y_j x_j \text{ for each } y = \{y_j\}_{j\geq 1} \in c_0(\mathbb{N})$$

is a bounded linear functional, every element ϕ in $(c_0(\mathbb{N}))^*$ is of the form φ_x for an element $x = \{x_j\}_{j\geq 1}$ in $\ell^1(\mathbb{N})$, and the assignment $x \mapsto \varphi_x$ is an isometric isomorphism of $\ell^1(\mathbb{N})$ to $(c_0(\mathbb{N}))^* = B(c_0(\mathbb{N}), \mathbb{C})$.

(b) (10 points) For this problem you may use (without proof) the special version of the principle of uniform boundedness called the Banach-Steinhaus theorem.

Theorem. Let X, Y be Banach spaces and suppose that $\{T_n\}_{n\geq 1}$ is a sequence in $B(X, Y)$ such that for each $x \in X$, $\{T_n x\}_{n\geq 1}$ is convergent in Y (and hence bounded). Then there is an operator T in $B(X, Y)$ such that $\lim_{n\to\infty} \|T_n x - Tx\| = 0$ for every $x \in X$.

Prove the following claim: for a sequence $\{x^n\}_n$ in $\ell^1(\mathbb{N})$ (thus $x^n = \{x^n_j\}_{j\geq 1} \in \ell^1(\mathbb{N})$ for each $n \geq 1$), we have $\lim_{n\to\infty} \sum_{j=1}^\infty x^n_j y_j = 0$ for all $y = \{y_j\}_{j\geq 1} \in c_0(\mathbb{N})$ if and only if $\sup \|x^n\|_1 < \infty$ and $\lim_{n\to\infty} x^n_j = 0$ for every $j \geq 1$.

Problem 5. Prove that there does not exist a sequence $\{\alpha_n\}_{n\geq 1} \subset (0, \infty)$ such that

$$\{x_n\}_{n\geq 1} \in \ell^1(\mathbb{N}) \iff \{\alpha_n x_n\}_{n\geq 1} \in \ell^\infty(\mathbb{N}). \tag{1}$$

Argue by contradiction, as follows. Assume that a sequence $\{\alpha_n\}_{n\geq 1} \subset (0, \infty)$ exists such that (1) holds.

(a) (5 points) Prove that $(Tx)_n = \alpha_n^{-1} x_n$ for $n \geq 1$ gives a well-defined lineal operator $T : \ell^\infty(\mathbb{N}) \to \ell^1(\mathbb{N})$. Prove that T is bounded.

(a) (5 points) Conclude that there exists a bounded inverse $T^{-1} : \ell^1(\mathbb{N}) \to \ell^\infty(\mathbb{N})$ for T. In particular, T is a homeomorphism. To reach a contradiction, you can for example use that the space

$$A = \{x = \{x_n\}_{n\geq 1} \mid \text{ there exists } N \in \mathbb{N} \text{ such that } x_n = 0, \forall n > N\}$$

of sequences with finitely many non-zero terms is dense in $\ell^1(\mathbb{N})$ but not in $\ell^\infty(\mathbb{N})$.