UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Eksamen i MAT4410 — Advanced Linear Analysis

Eksamensdag: 16. desember 2015

Tid for eksamen: 09.00 – 13.00

Oppgavesettet er på 2 sider.

Vedlegg: None

Tillatte hjelpemidler: None

Kontroller at oppgavesettet er komplett før
du begynner å besvare spørsmålene.

Oppgave 1

(**a**) (5p) Let λ be Lebesgue measure on the σ-algebra $\mathcal{B}$ of all Borel measurable sets on the half open interval $(0,1]$. We define a signed measure α on $\mathcal{B}$ by

$$\alpha(E) = \int_E \cos(2\pi x)\,\mathrm{d}\lambda(x), \quad E \in \mathcal{B}.$$

Find a Hahn-decomposition for α in $(0,1]$ and determine the positive and negative variations α^+ and α^- of α.

A measure β is given by

$$\beta(E) = \int_E \frac{1}{x}\,\mathrm{d}\lambda(x), \quad E \in \mathcal{B}.$$

(**b**) (15 p) Explain that $\lambda << \beta$, $|\alpha| << \beta$, and find the Radon-Nikodym derivatives

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\beta} \quad \text{and} \quad \frac{\mathrm{d}|\alpha|}{\mathrm{d}\beta}.$$

(**c**) (10p) Let μ be counting measure on $\mathcal{B}$ (that is, $\mu(E) =$ the number of elements of E if E is finite, $\mu(E) = +\infty$ if E is infinite). Show that $\lambda << \mu$. Prove that there is no $\mathcal{B}$-measurable function f on $(0,1]$ such that

$$\lambda(E) = \int_E f\,\mathrm{d}\mu, \quad E \in \mathcal{B}.$$

Explain why this does not contradict the Radon-Nikodym Theorem.

Oppgave 2

Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space (that is, $\mu(\Omega) < \infty$). Suppose that

(Fortsettes på side 2.)

(i) V is a closed subspace of $\mathcal{L}^p(\mu)$ for some $p,\ 1 \le p < \infty$,
and
(ii) V is contained in $\mathcal{L}^\infty(\mu)$.
In other words, V is a subspace of $\mathcal{L}^p(\mu) \cap \mathcal{L}^\infty(\mu)$, for some p, $1 \le p < \infty$, that is closed in the p-norm.

(**a**) (5p) Show that $V \subset \mathcal{L}^2(\mu)$ and that there is a constant C such that

$$(1) \qquad ||f||_2 \le C||f||_\infty \quad \text{whenever } f \in V.$$

Below we shall see that the inequality in (1) can be reversed.

(**b**) (15p) What do we mean by a closed linear map between two normed linear spaces? Show that the identity map

$$I \,:\, f \mapsto f,\ (V, ||\cdot||_p) \to (V, ||\cdot||_\infty)$$

is a closed map. Explain that there is an $M > 0$ such that

$$(2) \qquad ||f||_\infty \le M||f||_p\,, \quad \text{for all } f \in V.$$

(**c**) (15p) Assume next that $2 < p < \infty$ in (i) and (ii). Show that there is an $A > 0$ so that

$$||f||_\infty \le A||f||_2 \quad \text{whenever } f \in V.$$

Hint: $|f(x)|^p \le ||f||_\infty{}^{p-2}|f(x)|^2, \quad f \in V.$

(**d**) (10p) Suppose that $1 \le p \le 2$ in (i) and (ii). Show that there exists $B > 0$ such that

$$||f||_\infty \le B||f||_2$$

Hint: Consider $r = \frac{2}{p}$ and $s = \frac{2}{2-p}$. Then apply Hölder's inequality.

Oppgave 3

Let $\mathcal{M}$ be the σ-algebra of all Lebesgue measurable subsets of $\mathbb{R}$, λ be Lebesgue measure on $\mathcal{M}$. Assume that f is a continuous map from $[0,1]$ into $\mathbb{R}$ and consider the condition

$$(N) \qquad \lambda(f(E)) = 0 \text{ whenever } \lambda(E) = 0 \text{ and } E \subset [0,1].$$

(**a**) (15p) Suppose f satisfies (N). Show that $f(E) \in \mathcal{M}$ whenever $E \in \mathcal{M}$ and $E \subset [0,1]$.

(**b**) (10p) Show that if $f(E) \in \mathcal{M}$ whenever $E \in \mathcal{M}$ and $E \subset [0,1]$, then f satisfies (N). You can use (without proof) the fact that every $A \in \mathcal{M}$ for which $\lambda(A) > 0$ contains a nonmeasurable subset D with positive outer Lebesgue measure ($\lambda^*(D) > 0$).

THE END