Mandatory assignment in MAT4410 Fall 2018

The solution to the assignment must be submitted no later than Thursday, 25 October at 14:30 by using the electronic system Devilry (https://devilry.ifi.uio.no). You are expected to familiarise yourself with the rules for mandatory assignments available at uio.no/english/studies/examinations/compulsory-activities/mn-math-mandatory.html.

You must provide details for all your answers. The individual questions have different weights, as indicated. To pass the assignment you will need a score of at least 60 points.

Problem 1. Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}$ and let $\mathcal{B}_2$ be the Borel σ-algebra on $\mathbb{R}^2$. Suppose that μ and ν are σ-finite Borel measures on $\mathcal{B}$.

(a) (5 points) Prove that $A = \{(x, y) \mid x + y \in E\} \in \mathcal{B}_2$ for every $E \in \mathcal{B}$.

(b) (10 points) Prove that the functions $g(x) = \nu(E - x)$ and $h(y) = \mu(E - y)$ are Borel measurable for every $E \in \mathcal{B}$. If A is as in (a), prove that

$$(\mu \times \nu)(A) = \int_{\mathbb{R}} g(x) d\mu(x) = \int_{\mathbb{R}} h(y)) d\nu(y).$$

(c) (5 points) For $E \in \mathcal{B}$ we define $(\mu * \nu)(E) = \int_{\mathbb{R}} \mu(E - y) d\nu(y)$. (Note that $\mu * \nu = \nu * \mu$). Prove that $\mu * \nu$ is a Borel measure. (It is called the convolution of μ and ν.)

(d) (10 points) If $f \in \mathcal{L}^1(\mathbb{R}, \mathcal{B}, \mu * \nu)$, prove that

$$\int_{\mathbb{R}^2} f(x + y) d(\mu \times \nu)(x, y) = \int_{\mathbb{R}} f(t) d(\mu * \nu)(t).$$

(Hint: "bootstrap".)

(e) (10 points) Given a finite Borel measure μ, define the Fourier-Stieltjes transform of μ to be the function $\widehat{\mu} : \mathbb{R} \to \mathbb{C}$ given by

$$\widehat{\mu}(s) = \int_{\mathbb{R}} e^{its} d\mu(t),$$

for $s \in \mathbb{R}$. Show that this is well-defined and prove that if ν is another finite Borel measure then $\widehat{\mu * \nu}(s) = \widehat{\mu}(s)\widehat{\nu}(s)$ for $s \in \mathbb{R}$. (Thus the Fourier-Stieltjes transform takes convolution to pointwise product.)

Problem 2. (a) (5 points) Let ν be a complex measure on a measurable space $(\Omega, \mathcal{A})$. Let $|\nu|$ be its total variation and recall that $\nu = \nu_1^+ - \nu_1^- + i(\nu_2^+ - \nu_2^-)$ for unique positive, finite measures $\nu_1^+, \nu_1^-, \nu_2^+, \nu_2^-$. Show that

$$\frac{1}{\sqrt{2}}(\nu_1^+ + \nu_1^- + \nu_2^+ + \nu_2^-) \leq |\nu|.$$

(b) (5 points) Let λ be Lebesgue measure on $([0, 1], \mathcal{M}_{[0,1]})$ and define a complex measure by $\nu(A) = \lambda(A) + i\lambda(A)$ for every $A \in \mathcal{M}_{[0,1]}$. What can you say of the inequality in part (a)?

Problem 3. Let λ be Lebesgue measure on the σ-algebra $\mathcal{B}$ of Borel subsets of $[0,1)$. For $k \in \mathbb{N}$ put $T_k(x) = x + 2^{-k}$ for $x \in [0,1)$, where the addition is modulo 1.

(a) (5 points) Prove that $\mathcal{A}_k = \{A \in \mathcal{B} \mid T_k(A) = A\}$ is a σ-algebra of subsets of $[0,1)$ such that

$$\mathcal{A}_k = \Big\{ \bigcup_{j=0}^{2^k-1} T_k^j(A) \mid A \in \mathcal{B} \text{ and } A \subset [0, 2^{-k}) \Big\}.$$

(b) (5 points) Prove that a real valued function f defined on $[0,1)$ is $\mathcal{A}_k$ measurable if and only if f is Borel measurable and $f \circ T_k = f$.

(c) (10 points) If $f \in \mathcal{L}^1([0,1), \mathcal{B}, \lambda)$, show that the conditional expectation $\mathcal{E}(f \mid \mathcal{A}_k)$ satisfies $\mathcal{E}(f \mid \mathcal{A}_k) = g$ λ-a.e., where g is the function

$$g = \frac{1}{2^k} \sum_{j=0}^{2^k-1} f \circ T_k^j.$$

Note in particular that you have to argue that g is $\mathcal{A}_k$ measurable.

Problem 4. (10 points) Let T be a bounded linear operator between Banach spaces X and Y whose range $T(X)$ is closed in Y. Suppose that φ is a bounded linear functional on X, which we recall means that $\varphi : X \to \mathbb{C}$ is linear and bounded, having the property that

$$Tx = 0 \Rightarrow \varphi(x) = 0 \text{ for } x \in X;$$

note that this simply means that $\ker T \subset \ker \varphi$. Show that the assignment

$$\psi(Tx) = \varphi(x) \text{ for } x \in X$$

determines a well-defined bounded linear functional on $T(X)$.

Problem 5.

(a) (5 points) Let X, Y be normed spaces and $T : X \to Y$ a linear operator. Show that the graph of T is closed if and only if the following holds: whenever $\{x_n\}_n$ is a sequence in X with the property that x_n converges to 0 and $\{Tx_n\}_n$ is convergent in Y, then $\lim_{n\to\infty} Tx_n = 0$.

(b) (10 points) Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space. Suppose that $\phi : \Omega \to \mathbb{R}$ is an $\mathcal{A}$-measurable function such that

$$f \in L^2(\mu) \Rightarrow \phi f \in L^2(\mu),$$

where $L^2(\mu)$ is the Hilbert space of square integrable functions on Ω.

Show that the linear map $T : L^2(\mu) \to L^2(\mu)$ given by $Tf = \phi f$ is bounded. (Hint: use without proof that any sequence in $L^2(\mu)$ that converges to 0 admits a subsequence which converges $\mu-$a.e. to 0.)

(c) (5 points) Prove that ϕ must belong to $L^\infty(\mu)$, the Banach space of $\mathcal{A}$-measurable functions endowed with the essential supremum norm.