This note outlines a proof of Exercise 6.53 (a) and Exercise 6.54 (a) in [1] (se below for statements). We will use the following lemma, see Lemma 9.13 in [2].

Lemma 0.1. *Let $(\Gamma, \mathcal{S})$ and $(\Lambda, \mathcal{T})$ be measurable sets. If $\mathcal{S}_0$ is a nonempty collection of subsets of Γ that generates $\mathcal{S}$ and $\mathcal{T}_0$ is a nonempty collection of subsets of Λ that generates $\mathcal{T}$, show that the collection $\mathcal{S}_0 \times \mathcal{T}_0$ consisting of cartesian products $B \times C$ with $B \in \mathcal{S}_0$ and $C \in \mathcal{T}_0$ generates the product σ-algebra $\mathcal{S} \times \mathcal{T}$.*

Proof. Denote $\mathcal{A}$ the σ-algebra generated by the collection $\mathcal{S}_0 \times \mathcal{T}_0$. Let $\pi_1 : \Gamma \times \Lambda \to \Gamma$ be the projection onto the first coordinate $\pi_1(x, y) = y$. Verify directly that the set $\{A \subseteq \Gamma \mid \pi^{-1}(A) \in \mathcal{A}\}$ is a σ-algebra on Γ which contains $\mathcal{S}_0$ (use that the inverse image of a map preserves taking complements of sets and unions of sets). Then this set contains $\mathcal{S}$. It follows that the collection $\mathcal{S} \times \Lambda$ consisting of cartesian products $B \times \Lambda$ with $B \in \mathcal{S}$ is contained in $\mathcal{A}$. Similarly, $\Gamma \times \mathcal{T} \subset \mathcal{A}$. Then

$$(\mathcal{S} \times \Lambda) \cap (\Gamma \times \mathcal{T}) = (\mathcal{S} \cap \Gamma) \times (\mathcal{T} \cap \Lambda) = \mathcal{S} \subset \mathcal{A}.$$

This shows that the collection of rectangles $\mathcal{U} = \{B \times C \mid B \in \mathcal{S}\, C \in \mathcal{T}$ is contained in $\mathcal{A}$, hence the product σ-algebra, being generated by these rectangles, is contained in $\mathcal{A}$.

The other inclusion follows because $\mathcal{S}_0 \times \mathcal{T}_0$ is contained in the collection $\mathcal{U}$. □

Exercise 6.53(a) in [1] For $n \geq 1$ let $\mathcal{B}_n$ be the Borel algebra on $\mathbb{R}^n$, i.e. the σ-algebra generated by the open subsets of $\mathbb{R}^n$. Then $\mathcal{B}_2 = \mathcal{B} \times \mathcal{B}$.

To prove this, use that $\mathcal{B}$ is generated by the collection $\mathcal{E}$ of open sets in $\mathbb{R}$, so Lemma 0.1 implies $\mathcal{B} \times \mathcal{B} = \mathcal{B}(\mathcal{E} \times \mathcal{E})$. Since every set in $\mathcal{E} \times \mathcal{E}$ is open in $\mathbb{R}^2$, we get $\mathcal{B}(\mathcal{E} \times \mathcal{E}) \subset \mathcal{B}_2$. For the converse inclusion, use that every open ball in $\mathbb{R}^2$ can be written as a countable union of elements in $\mathcal{E} \times \mathcal{E}$, in fact as a countable union of sets of the form $(a, b) \times (c, d)$ with a, b, c, d rational numbers. Then we get $\mathcal{B}_2 \subset \mathcal{B}(\mathcal{E} \times \mathcal{E})$ and the exercise follows.

Lemma 0.2. *Let $(\Omega_j, \mathcal{A}_j, \mu_j)$, $j = 1, 2, 3$ be σ-finite measure spaces. Then $(\mathcal{A}_1 \times \mathcal{A}_2) \times \mathcal{A}_3 = \mathcal{A}_1 \times (\mathcal{A}_2 \times \mathcal{A}_3)$ and $(\mu_1 \times \mu_2) \times \mu_3 = \mu_1 \times (\mu_2 \times \mu_3)$.*

Proof. For the first claim, use rectangles $(A_1 \times A_2) \times A_3 = A_1 \times (A_2 \times A_3)$ for $i = 1, 2, 3$ as generating sets and apply the previous lemma. For the second, use that $(\mu_1 \times \mu_2) \times \mu_3$ and $\mu_1 \times (\mu_2 \times \mu_3)$ coincide on rectangles $A_1 \times A_2 \times A_3$ generating $\mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3$, and apply the result about existence and uniqueness of product measure. □

Note that Lemma 0.2 allows us to define $\mathcal{B} \times \cdots \times \mathcal{B}$ on $\mathbb{R} \times \cdots \times \mathbb{R}$, where we have n factors. One can similarly to the above prove that $\mathcal{B}_n = \mathcal{B} \times \cdots \times \mathcal{B}$.

Exercise 6.54(a) [1] Show that $\mathcal{B}_2$ is also generated by the 2-dimensional intervals $I \times J$ of $\mathbb{R}^2$, where I, J are intervals of form $(a, b]$ or (c, ∞) in $\mathbb{R}$.

To prove this note that intervals of the form $(a, b]$ for $-\infty \leq a \leq b < \infty$ can be expressed as countable intersections of intervals $(a, b + 1/n)$ for $n \in \mathbb{N}$. Thus we can reduce to open sets of the form $(a, b) \times (c, d)$ and use the previous exercise.

References

[1] J.N. McDonald and N.A. Weiss, A course in Real Analysis, 2nd edition, Academic Press, Amsterdam, 2013.

[2] G. Teschl, Topics in real and functional analysis, http://www.mat.univie.ac.at/ gerald/ftp/book-fa/index.html.