This note outlines two (very similar) proofs of Exercise 14.5 in J.N. McDonald and N.A. Weiss, A course in Real Analysis, 2nd edition, Academic Press, Amsterdam, 2013, stating that every Hilbert space is reflexive as a Banach space.

Let H be a Hilbert space and consider the map $J : H \to H^{**}$ defined by

$$\langle \varphi, J(y)\rangle = \langle y, \varphi\rangle = \varphi(y)$$

for $y \in H$ and $\varphi \in H$. We claim that J is onto (surjective).

Recall that by the Riesz lemma, a functional ℓ on H is of the form $\ell = \varphi_y$ with $\varphi_y(x) = (x \mid y)$ for all $x \in H$ for a unique vector $y \in H$ such that $\|\ell\| = \|y\|$. There is therefore a bijective map $\Phi : H \to H^*$ defined by $\Phi(y) = \varphi_y$ for $y \in H$.

Proof 1, almost as done in class: Let $\ell \in H^{**} = (H^*)^*$, thus $\ell : H^* \to \mathbb{C}$ is a bounded linear functional. The composition $\ell \circ \Phi : H \to \mathbb{C}$ satisfies $(\ell \circ \Phi)(y + z) = (\ell \circ \Phi)(y) + (\ell \circ \Phi)(z)$ and $(\ell \circ \Phi)(\alpha y) = \overline{\alpha}(\ell \circ \Phi)(y)$ for all $y, z \in H$ and $\alpha \in \mathbb{C}$. Then $\tilde{\ell} : H \to \mathbb{C}$ given by $\tilde{\ell}(x) = \overline{\ell(\Phi(x))}$ for $x \in H$ is a bounded linear functional on H. By the Riesz lemma, there is a unique $y \in H$ such that $\tilde{\ell}(x) = (x \mid y)$ for all $x \in H$. We claim that $J(y) = \ell$, which will imply the surjectivity claim. It suffices to show that $J(y)(\Phi(x)) = \ell(\Phi(x))$ for all $\Phi(x) \in H^*$, where $x \in H$. For $x \in H$ we have

$$\langle \Phi(x), J(y)\rangle = \varphi_x(y) = (y \mid x),$$

and

$$\langle \Phi(x), \ell\rangle = \ell(\Phi(x)) = \overline{(x \mid y)} = (y \mid x),$$

as needed.

Proof 2: It is immediate from the construction that $\Phi(y_1 + y_2) = \varphi(y_1) + \varphi(y_2)$ for $y_1, y_2 \in H$. For $\alpha \in \mathbb{C}$, we have $\Phi(\alpha y) = \varphi_{\alpha y} = \overline{\alpha}\varphi_y$, so Φ is a conjugate linear bijective map. With this map we can identify H^* with H as a Hilbert space, where we take the inner product

$$(\Phi(y_1) \mid \Phi(y_2)) = (y_2 \mid y_1) \text{ for } y_1, y_2 \in H.$$

The inner product on H^* is linear in the first variable and conjugate linear in the second variable, as can be seen from the following computations:

$$\begin{aligned}\big(\alpha\Phi(y_1) \mid \Phi(y_2)\big) &= \big(\Phi(\overline{\alpha}y_1) \mid \Phi(y_2)\big) = \big(y_2 \mid \overline{\alpha}y_1\big)\\ &= \alpha\big(y_2 \mid y_1\big) = \alpha\big(\Phi(y_1) \mid \Phi(y_2)\big)\end{aligned}$$

and

$$\begin{aligned}\big(\Phi(y_1) \mid \alpha\Phi(y_2)\big) &= \big(\Phi(y_1) \mid \Phi(\overline{\alpha}y_2)\big) = \big(\overline{\alpha}y_2 \mid y_1\big)\\ &= \overline{\alpha}\big(y_2 \mid y_1\big) = \overline{\alpha}\big(\Phi(y_1) \mid \Phi(y_2)\big).\end{aligned}$$

Now let $\ell \in H^{**} = (H^*)^*$, so by the Riesz lemma applied to H^* there is $\Phi(y) \in H^*$ such that

$$\ell(\Phi(x)) = \big(\Phi(x) \mid \Phi(y)\big) \text{ for all } \Phi(x) \in H^*.$$

Then $\langle \Phi(x), J(y)\rangle = \Phi(x)(y) = \varphi_x(y) = (y \mid x)$ and $\langle \Phi(x), \ell\rangle = \ell(\Phi(x)) = \big(\Phi(x) \mid \Phi(y)\big) = (y \mid x)$, which implies $J(y) = \ell$.