MAT4410 20.11.2018 (1)

The Fourier transform

For $f \in L^1(\mathbb{R})$ define $\hat{f}: \mathbb{R} \to \mathbb{C}$ by

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-itx} dx \quad , t \in \mathbb{R}.$$

<u>Lemma</u>: For every $f \in L^1(\mathbb{R})$ we have $\hat{f}: \mathbb{R} \to \mathbb{C}$ is continuous and bd, with $\|\hat{f}\|_\infty \leq \|f\|_1 \cdot \frac{1}{\sqrt{2\pi}}$

<u>Proof</u>: From $|\hat{f}(t)| \leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |f(x) e^{-itx}| dx$ we get the claim on bd.
For continuity, need:

<u>Proposition</u> Let $(\Omega, \mathcal{A}, \mu)$ be a (σ-finite) measure space and $I \subseteq \mathbb{R}$ interval. Suppose that:

(a) for each $t \in I$, the function $x \mapsto h(x,t)$ is in $L^1(\Omega)$ and
(b) for each $x \in \Omega$, the function $t \mapsto h(x,t)$ is continuous at some $t_0 \in I$.

If there is $g: \Omega \to [0, \infty]$ integrable s.t.

$$\forall x \in \Omega, \forall t \in I: \quad |h(x,t)| \leq g(x), \text{ ~~then~~}$$

$t \longmapsto \int_\Omega h(x,t) d\mu(x)$ is continuous at t_0. !

so $h(x,t_n) \to h(x,t), \forall x$

<u>Proof</u>: exercise (use DCT, after picking $\{t_n\} \subset I$ s.t. $t_n \to t_0$).

Back to lemma, apply the proposition to $\underline{h(x,t) = f(x) e^{-itx}}$ on $\mathbb{R} \times \mathbb{R}$.

In fact, $\hat{f} \in C^\infty(\mathbb{R})$ i.e. continuously differentiable of every order k, $k \geq 1$, at every point $t \in \mathbb{R}$.

Need lemma.

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Definition The Schwartz class (or rapidly decreasing functions)

$\mathcal{S}(\mathbb{R})$ consists of $f \in C^\infty(\mathbb{R})$ s.t. (1) $\sup_{x\in\mathbb{R}} |x^N f^{(k)}(x)| < \infty$, for all $k, N \in \mathbb{Z}_+$. ↑ partial derivative

(1) $\Leftrightarrow$ for every $k, N \in \mathbb{Z}_+$ $\exists\, C_{k,N} \geq 0$ s.t. $|x|^N \cdot |f^{(k)}(x)| \leq C_{k,N}$, $\forall x \in \mathbb{R}$

(1) $\Longleftrightarrow$ (2): $(1+|x|)^N |f^{(k)}(x)| \leq C_{k,N}$, $\forall x \in \mathbb{R}$

$\Leftarrow$ Clear since $|x|^N |f^{(k)}(x)| \leq (1+|x|)^N |f^{k}(x)|$

$\Rightarrow$ Since $(1+|x|)^N = \mathrm{Pol}(|x|)$ and each $|x|^j |f^{(k)}(x)|$ is bd.

Fact $\mathcal{S}(\mathbb{R}) \subseteq \mathcal{L}^1(\mathbb{R})$: $\int_{\mathbb{R}} |f(x)| dx < C_{1,0} < \infty$.
even $\mathcal{L}^p(\mathbb{R})$

Example (1) $C_c^\infty = \{ f: \mathbb{R} \to \mathbb{C} \mid f^{(k)}$ exists and is cont, $\forall k$, and $\operatorname{supp} f = \overline{\{x \mid f(x) \neq 0\}}$ is compact$\}$

Suppose $f \in C_c^\infty$, $\operatorname{supp}(f) \subseteq [-M, M]$, then clearly $\operatorname{supp} f^{(k)} \subseteq [-M, M]$.

Since $f^{(k)}$ is cont on $[-M, M]$, it is bd so for $\forall k, N \in \mathbb{Z}_+$

$\exists\, C_{k,N} \geq 0$ st $|f^{(k)}(x)| \leq \dfrac{C_{k,N}}{(1+M)^N}$, thus

$$(1+|x|)^N |f^{(k)}(x)| \leq (1+M)^N |f^{(k)}(x)| \leq C_{k,N} \text{ so } f \in \mathcal{S}(\mathbb{R})$$

Example 2

$\varphi(x) = e^{-x^2}$

$$\left| x^N \frac{d^k}{dx^k} e(-x^2) \right| \leq \left| x^N \sum_{l=0}^{k} C_l x^l e(-x^2) \right| \leq \sum_{l=0}^{k} C_l |x^{N+l} e(-x^2)|,$$

which is uniformly bd since $|x|^{N+l} e(-x^2) \to 0$ as $|x| \to \infty$ and $x^{N+l} e(-x^2)$ is continuous, thus bd.

We get $\varphi \in \mathcal{S}(\mathbb{R})$ and $\operatorname{supp}(f)$ not compact.

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Lemma (Thm 15.9(a) and (e) in [MW]).
Let $f \in \mathcal{S}(\mathbb{R})$. Then for $n \in \mathbb{N}$ we have:

(A) $\widehat{f^{(n)}}(t) = (it)^n \hat{f}(t)$.

(B) $\widehat{x^n f}(t) = \left(i\frac{d}{dt}\right)^n \hat{f}(t)$ where $x^n f$ is the function $x \mapsto x^n f(x)$.
Hence, $\hat{f} \in \mathcal{S}(\mathbb{R})$.

Proof: Induction over $n \geq 1$.

(A) $n=1$:

$$\sqrt{2\pi}\, it\, \hat{f}(t) = it \int_{\mathbb{R}} f(x) e^{-itx} dx = -\int_{\mathbb{R}} \left(\frac{d}{dx} e^{-itx}\right) f(x)\, dx$$

$$= -\lim_{k\to\infty} \int_{-k}^{k} \left(\frac{d}{dx} e^{-itx}\right) f(x)\, dx$$

$$\overset{\substack{\text{derivation}\\ \text{by parts}\\ (f \text{ cont})}}{=} -\lim_{k\to\infty} \left(\left[-e^{-itx} f(x)\right]_{-k}^{k} + \int_{-k}^{k} f'(x) e^{-itx} dx \right)$$

$$= \lim_{k\to\infty} \left[-f(k) e^{-itk} + f(-k) e^{itk}\right] + \lim_{k} \int_{-k}^{k} f'(x) e^{-itx} dx$$

Now $|f(x)| \leq C_{1,N} |x|^{-N}$ for $N \geq 1$ so $f(x) \to 0$ as $|x| \to \infty$,

Thus $\sqrt{2\pi}\, it\, \hat{f}(t) = \int_{\mathbb{R}} f'(x) e^{-itx} dx$ so $\boxed{it \hat{f}(t) = \widehat{f'}(t).}$

Induction shows $n \mapsto n+1$.

For (B), at $n=1$ need: $\widehat{xf}(t) = \left(i \frac{d}{dt}\right) \hat{f}(t) \overset{\text{i.e.}}{=} i \hat{f}'(t)$.

So, need $\int_{\mathbb{R}} x f(x) e^{-itx} dx = i \frac{d}{dt} \left(\int_{\mathbb{R}} f(x) e^{-itx} dx \right)$

By DCT, since $t \mapsto f(x) e^{-itx}$ is in $C^1(\mathbb{R})$, $\forall x$, and
$\frac{d}{dt}\left(f(x) e^{-itx}\right) = -ix f(x) e^{-itx}$ is bd by $x f(x) \in \mathcal{L}^1(\mathbb{R})$,

get $i \frac{d}{dt} \int_{\mathbb{R}} f(x) e^{-itx} dx = i \int_{\mathbb{R}} \frac{\partial}{\partial t}\left(f(x) e^{-itx}\right) dx = i \int (-ix) f(x) e^{-itx} = \widehat{xf}(t)$.

→ [MW, 5.78] or [T, 9.14] ⇝ Induction

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From (A) and (B) we can deduce that $\hat{f} \in \mathcal{S}(\mathbb{R})$ whenever $f \in \mathcal{S}(\mathbb{R})$.

By (B), since $\widehat{x^n f}$ cont, $\forall n \geq 1$, we have $\left(\frac{i\,d}{dt}\right)^n \hat{f}(t)$ cont.,

thus $\hat{f} \in C^\infty(\mathbb{R})$. For k, N need $|t|^n \left|\frac{d^k}{dt^k} \hat{f}(t)\right|$ to be

a bounded function. For all $N \geq 0$, $f^{(N)} \in \mathcal{S}(\mathbb{R})$ by def of $\mathcal{S}(\mathbb{R})$
so the fact that $\|\hat{f}\|_u \leq cst \|f\|_1$ implies $\widehat{f^{(N)}}$ is bd, so by (A)
also $|t^N \hat{f}(t)|$ is bd. Use this with $\frac{d^k}{dt^k}\hat{f}$ in place of $\hat{f}$
according to (B).

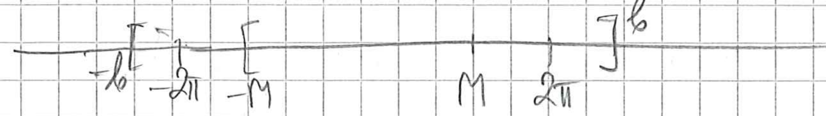
<u>Theorem</u> Let $f \in C_c^\infty(\mathbb{R})$, then $f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(t) e^{itx} dt$, $\forall x \in \mathbb{R}$.

<u>Proof</u>: Since $f \in C_c^\infty(\mathbb{R})$, we have $f \in \mathcal{S}(\mathbb{R})$.

By lemma, $\hat{f} \in \mathcal{S}(\mathbb{R})$. Thus for each $N = 0, 1, 2, \dots$, there is $C_N \geq 0$ s.t.

$$(\#) \quad |\hat{f}(t)| \leq \frac{C_N}{(1+|t|)^N} \text{ for all } t \in \mathbb{R}$$

Let $\operatorname{supp} f \subseteq [-M, M]$ for some $M > 0$. Let $b \in \mathbb{R}$ s.t. $b \geq \max\{2\pi, M\}$.



So $f(x) = 0$, $|x| \geq b$, where $b > 0$.

Consider $L^2[-b, b]$ with inner product:

$$\langle f, g \rangle = \int_{-b}^{b} f \bar{g} \, dx.$$

$\frac{\pi}{b} \leq \frac{1}{2}$

<u>Fact</u>: $\left\{ \frac{1}{\sqrt{2b}} e^{ik\frac{\pi}{b}s} = e_k^b(s) \right\}_{k \in \mathbb{Z}}$ is an o.n.b for $L^2[-b, b]$.

<u>Proof</u>: see Rynne–Youngson or note that $U : L^2[-\pi, \pi] \to L^2[-b, b]$,

$(Ug)(s) = \sqrt{\frac{\pi}{b}}\, g\left(\frac{\pi}{b} s\right)$ is a unitary operator, so it takes the o.n.b $\left\{ \frac{1}{\sqrt{2\pi}} e^{ikx} \right\}$ to the o.n.b $\sqrt{\frac{\pi}{b}} \cdot \frac{1}{\sqrt{2\pi}} e^{ik\frac{\pi}{b}x} = \frac{1}{\sqrt{2b}} e^{ik\frac{\pi}{b}x}$.

$f|_{[-b,b]}$ has a Fourier series in the o.n.b $\{e_k^b\}_{k \in \mathbb{Z}}$, with the n'th coefficient:

$$\widehat{f|_{[-b,b]}}(n) = \frac{1}{\sqrt{2b}} \int_{-b}^{b} f(y) e^{-in\frac{\pi}{b}y} dy.$$

Since $f(y) = 0$ for $|y| > b$, we get

Usual Fourier coeff at $n \in \mathbb{Z}$ $\longrightarrow$ $\widehat{f|_{[-b,b]}}(n) = \frac{1}{\sqrt{2b}} \int_{\mathbb{R}} f(y) e^{-in\frac{\pi}{b}y} dy = \frac{\sqrt{2\pi}}{\sqrt{2b}} \hat{f}\left(n\frac{\pi}{b}\right)$. $\longleftarrow$ Fourier coeff at point in $\mathbb{R}$.

Thus the Fourier series for $f|_{[-b,b]}$ is $\frac{1}{\sqrt{2b}} \sum_{n \in \mathbb{Z}} \widehat{f|_{[-b,b]}}(n)\, e^{in\frac{\pi}{b}x}$, i.e.

7) Since $f \in C^1[-b,b]$, we have pointwise convergence. ⑤

$$f(x) = \sqrt{2\pi} \sum_{n\in\mathbb{Z}} \frac{1}{2b} \hat{f}\left(n\frac{\pi}{b}\right) e^{in\frac{\pi}{b}x}, \quad \forall x \in (-b,b).$$

Idea: express RHS as $\sum_n \int_{\mathbb{R}} a_n \chi_{I_n}(t)\,dt$ to approximate it with $\int_{\mathbb{R}} \hat{f}(x)e^{itx}dx$.

I_n interval

For $b>0$ and each $x\in\mathbb{R}$ define $\boxed{g_{b,x}(t) = \hat{f}\left(n\frac{\pi}{b}\right)e^{in\frac{\pi}{b}x}}$; $g:\mathbb{R}\to\mathbb{C}$.
(fixed by arbitrary)

Partition of $\mathbb{R}$ with $n\frac{\pi}{b} \le t < (n+1)\frac{\pi}{b}$ (intervals of length $\frac{\pi}{b}$).

$$\Rightarrow f(x) = \sqrt{2\pi}\sum_{n\in\mathbb{Z}} \frac{1}{2b}\cdot\frac{b}{\pi}\int_{[n\frac{\pi}{b},(n+1)\frac{\pi}{b})} g_{b,x}(t)\,dt = \frac{1}{\sqrt{2\pi}}\int_{\bigcup_{n\in\mathbb{Z}}[n\frac{\pi}{b},(n+1)\frac{\pi}{b})} g_{b,x}(t)\,dt,$$

so $f(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} g_{b,x}(t)\,dt$ for $x\in(-b,b)$, where $g_{b,x}$ is integrable because

$$|g_{b,x}(t)| = \left|\hat{f}\left(n\frac{\pi}{b}\right)\right|\cdot\left|e^{in\frac{\pi}{b}x}\right| = \left|\hat{f}\left(n\frac{\pi}{b}\right)\right| \overset{(\#)}{\le} \frac{C_N}{\left(1+\left|n\frac{\pi}{b}\right|\right)^N},$$

however $n\frac{\pi}{b} \le t < n\frac{\pi}{b}+\frac{1}{2}$ so $|t|-\frac{1}{2} < \left|n\frac{\pi}{b}\right|$ so $|t|+\frac{1}{2} \le \left|n\frac{\pi}{b}\right|+1$,

thus $|g_{b,x}(t)| \le \frac{C_N}{(1/2+|t|)^N} \in L^1(\mathbb{R})$ for $N\ge 2$.

By continuity of $\hat{f}$, for each $t\in\mathbb{R}$, $\lim_{b\to\infty} g_{b,x}(t) = \hat{f}(t)e^{itx}$, as x is fixed, since the interval $[n\frac{\pi}{b},(n+1)\frac{\pi}{b})$ degenerates to the point $\{t\}$.

By DCT (use sequence $b_m\to\infty$) get $f(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\hat{f}(t)e^{itx}\,dt$.
and majorant $\frac{C_N}{(1/2+|t|)^N}$

$\boxtimes$.