MAT4410 2018: Suggested solution

Problem 1a. Solution: Show first that $\sigma(\mathcal{C}) = \{\bigcup_{E \in J} E \mid J \subset \mathcal{C}\}$, by proving that the set on the right-hand side of the equality is a σ-algebra containing every element of $\mathcal{C}$. Let f be $\sigma(\mathcal{C})$-measurable and $E \in \mathcal{C}$. Show that $f|_E$ is constant. If this were not the case, let $x, y \in E$ such that $f(x) = s \neq t = f(y)$. Pick disjoint open subsets U and V of $\mathbb{R}$ with $s \in U$ and $t \in V$. Now $f^{-1}(U)$ is in $\sigma(\mathcal{C})$ and has non-empty intersection with E, hence it contains E by the form of elements in $\sigma(\mathcal{C})$. Similarly, $f^{-1}(U)$ contains E. Then $f(E)$ is contained in $U \cap V$, a contradiction. Thus for each $E \in \mathcal{C}$ there is $a_E \in \mathbb{R}$ such that $f(x) = a_E$ for all $x \in E$ and the claim about f follows.

Problem 1b. Apply 1a to get $\mathcal{E}(f \mid \mathcal{G})$ of the claimed form. For each $j \geq 1$,

$$\int_{F_j} f dP = \int_{F_j} \mathcal{E}(f \mid \mathcal{G}) dP = \int_{F_j} \sum_{k=1}^{n} a_k \chi_{F_k} dP = \sum_{k=1}^{n} a_k \int_{F_j} \chi_{F_k} dP = a_j P(F_j),$$

where we used that $\int_{F_j} \chi_{F_k} dP = 0$ whenever $k \neq j$, thus showing that only one term in the sum is non-zero. Thus $a_j = P(F_j)^{-1} \int_{F_j} f dP$ for each j.

Problem 2a Given $(\Omega, \mathcal{A}, \mu)$ a σ-finite measure space, $(\mathbb{R}, \mathcal{M}, \lambda)$ and $f : \Omega \to [0, \infty)$ an $\mathcal{A}$ measurable function, define $g : \Omega \times \mathbb{R} \to \mathbb{R}$ by $g(x, t) = f(x) - t$. To first claim is that g is measurable with respect to the product σ-algebra $\mathcal{A} \times \mathcal{M}$ on $\Omega \times \mathbb{R}$. One solution is: use that $g(x, t) = f(x) - h(t)$, $h(t) =$ for $t \in \mathbb{R}$, so g is the difference of two $\mathcal{A} \times \mathcal{M}$-measurable functions, and is therefore measurable. Another solution is to let $s \in \mathbb{R}$ and verify directly that

$$g^{-1}((s, \infty)) = \bigcup_{r \in \mathbb{Q}} f^{-1}((s + r, \infty)) \times (-\infty, r),$$

which is a countable union of measurable rectangles, and therefore in $\mathcal{A} \times \mathcal{M}$. The equality of sets is verified by proving the two inclusions. For example, if $(x, t) \in g^{-1}((s, \infty))$, then $f(x) > t + s$, so there is $r_x \in \mathbb{Q}$ such that $f(x) - s > r_x > t$, which shows $(x, t) \in f^{-1}((s + r_x, \infty)) \times (-\infty, r_x)$, a set in the countable union. The other inclusion is easy.

A third possibility is to use the bootstrapping technique: Suppose first that $f = \chi_E$ for $E \in \mathcal{A}$. Then

$$g(x,t) = \begin{cases} 1-t, & \text{if } x \in E \\ -t, & \text{if } x \in E^c. \end{cases}$$

Let $r \in \mathbb{R}$. Then $g^{-1}((r,\infty)) = \{(x,t) \mid g(x,t) > r\}$, so $g^{-1}((r,\infty))$ is the union of the sets $E \times (-\infty, 1-r)$ and $E^c \times (-\infty, -r)$, each of which is a measurable rectangle in $\mathcal{A} \times \mathcal{M}$, showing that $g^{-1}((r,\infty)) \in \mathcal{A} \times \mathcal{M}$.

Suppose next that f is a simple function in standard form, so $f = \sum_{i=1}^n a_i \chi_{E_i}$ with $E_1, \ldots, E_n$ pairwise disjoint in $\mathcal{A}$. Then once can check that $g^{-1}((s,\infty))$ is a union of sets $E_i \times (-\infty, -a_i - r)$, each of which is in $\mathcal{A} \times \mathcal{M}$.

Finally, let f be arbitrary nonnegative $\mathcal{A}$-measurable. There exists a nonincreasing sequence $\{s_n\}$ of nonnegative, measurable functions such that s_n converges to f pointwise, μ-a.e. We have

$$g^{-1}((r,\infty)) = \{(x,t) \mid \lim_n s_n(x) > r + t\},$$

so, by approximating each t with a rational number t_m, we get

$$g^{-1}((r,\infty))) = \bigcup_{n,m} \{(x,t) \mid s_n(x) \geq r + t_m\},$$

which by the previous steps of the problem is in $\mathcal{A} \times \mathcal{M}$.

For the second part of 2a, let $A \in \mathcal{A}$ and define

$$\begin{aligned} B &= \{(x,t) \in \Omega \times \mathbb{R} \mid 0 \leq t \leq f(x), x \in A\}, \\ C &= \{(x,t)\Omega \times \mathbb{R} \mid t = f(x), x \in A\}. \end{aligned}$$

To show that $B, C \in \mathcal{A} \times \mathcal{M}$, note that $B = (A \times [0,\infty)) \cap g^{-1}([0,\infty))$, which is in $\mathcal{A} \times \mathcal{M}$ by the measurability of g. Similarly, $C = (A \times [0,\infty)) \cap g^{-1}(\{0\})$, again in $\mathcal{A} \times \mathcal{M}$.

Problem 2b. To show that $(\mu \times \lambda)(B) = \int_A f\, d\mu$ and $(\mu \times \lambda)(C) = 0$, note that the x section of B is $B_x = \{t \in \mathbb{R} \mid 0 \leq t \leq f(x), x \in A\}$. By Tonelli's theorem, which applies because both $(\Omega, \mathcal{A}, \mu)$ and $(\mathbb{R}, \mathcal{M}, \lambda)$ are σ-finite, we obtain

$$(\mu \times \lambda)(B) = \int_{\Omega \times \mathbb{R}} \chi_B \, d(\mu \times \lambda) = \int_\Omega \lambda(B_x) d\mu(x).$$

This gives

$$(\mu \times \lambda)(B) = \int_\Omega \lambda([0, f(x)])\chi_A(x) d\mu(x) = \int_A f(x) d\mu(x).$$

Since $C_x = \{t \mid g(x,t) = 0, x \in A\}$, Tonelli's theorem gives

$$(\mu \times \lambda)(C) = \int_\Omega \lambda(C_x) d\mu(x) = \int_A 0 \, d\mu = 0.$$

Problem 3 One solution is to apply the Hahn-Banach theorem to the linear functional φ to obtain a linear functional $\Phi : L^p(\mu) \to \mathbb{C}$ such that Φ restricted to $\mathcal{S}$ is φ and $\|\Phi\| = \|\varphi\| < \infty$. Thus, Φ is a bounded linear functional on $L^p(\mu)$ and since $(L^p(\mu))^* \cong L^q(\mu)$ there is a function $g \in L^q(\mu)$ such that

$$\Phi(f) = \int_\Omega fg \, d\mu \text{ for all } f \in L^p(\mu),$$

thus in particular for all f in $\mathcal{S}$. Another solution is to use that the space $\mathcal{S}$ is dense in $L^p(\mu)$, a fact proved in class, and to use the existence of a unique extension of ϕ from $\mathcal{S}$ to $L^p(\mu)$ with same norm. Then use the duality $(L^p(\mu))^* \cong L^q(\mu)$ to find the required g.

Problem 4a. State PUB for a sequence $\{A_n\}$ of bounded linear operators between normed spaces.

Problem 4b. We are given a sequence A_n of operators in $B(\Omega, \Lambda)$ for Banach spaces Ω, Λ such that for each $y \in \Omega$ the sequence $\{A_n y\}_n$ converges in Λ. To prove that $A : \Omega \to \Lambda$ given by $Ay = \lim_n (A_n y)$ for each $y \in \Omega$ is linear and bounded, note first that if y_1 and y_2 are given in Ω, then we have convergent sequences $\{A_n(y_1)\}_n$, $\{A_n(y_2)\}_n$ and $\{A_n(y_1 + y_2)\}_n$ in Λ. Since $A_n(y_1 + y_2) = A_n(y_1) + A_n(y_2)$ for every $n \geq 1$ and since the limit of a convergent sequence in a normed space is unique, we must have $A(y_1 + y_2) = A(y_1) + A(y_2)$. A similar argument (write it down) shows that $A(\alpha y) = \alpha A y$ for $y \in \Omega$ and $\alpha \in \mathbb{C}$, showing that A is linear. To see that it is bounded, note that for each $y \in \Omega$, the sequence $\{A_n y\}_n$ is bounded, being convergent, so the family $\{A_n\}_n$ is pointwise bounded. By PUB, $\{A_n\}_n$ is uniformly bounded, meaning that $\sup_{n \geq 1} \|A_n\| < \infty$. By definition of A we have $\|A\| \leq \sup_{n \geq 1} \|A_n\|$, which implies that A is bounded.

Let $c_0(\mathbb{N})$ denote the Banach space of sequences converging to zero considered with the supremum norm $\|\cdot\|_\infty$. Suppose that $x = \{x_j\}_{j \geq 1}$ is a sequence of complex numbers such that $\sum_{j=1}^\infty x_j y_j$ is a convergent series for every $\{y_j\}_{j \geq 1} \in c_0(\mathbb{N})$. For every $n \geq 1$, define $\phi_n : c_0(\mathbb{N}) \to \mathbb{C}$ by $\phi_n(y) = \sum_{j=1}^n x_j y_j$ for $y = \{y_j\}_{j \geq 1}$ in $c_0(\mathbb{N})$.

Problem 4c. Let $n \geq 1$. Since $y_n \to 0$, the sequence $y = \{y_n\}_{n \geq 1}$ is bounded. Then

$$|\phi_n(y)| \leq \sum_{j=1}^{n} |x_j y_j| \leq \|y\|_\infty \sum_{j=1}^{n} |x_j|.$$

Hence $\|\phi_n\| \leq \sum_{j=1}^n |x_j|$, which is a constant (since the sum is finite) so ϕ_n is bounded. Routine verifications show that ϕ_n is linear (fill them in).

Problem 4d. To show that $x \in l^1(\mathbb{N})$, let $y = \{y_n\}_{n\geq 1}$ in $c_0(\mathbb{N})$. Then

$$\lim_{n\to\infty} \phi_n(y) = \lim_{n\to\infty} \sum_{j=1}^{n} x_j y_j,$$

which is dominated by the convergent sum $\sum_{j=1}^\infty x_j y_j$. Thus by (4b), the map $\phi(y) = \lim_{n\to\infty} \phi_n(y)$ defines a bounded functional $\phi : c_0(\mathbb{N}) \to \mathbb{C}$ with norm dominated by $\sup\{\|\phi_n\| \mid n \in \mathbb{N}\} < \infty$. For every $j \geq 1$ let $\alpha_j \in \mathbb{C}$ such that $x_j \alpha_j = |x_j|$. Since

$$\phi_n(\alpha_1, \ldots, \alpha_n, 0 \ldots) = \sum_{j=1}^{n} x_j \alpha_j = \sum_{j=1}^{n} |x_j|,$$

it follows that $\|\phi_n\| = \sum_{j=1}^n |x_j|$ for $n \geq 1$. Hence $\|x\|_1 = \sum_{n=1}^\infty |x_n| = \sup\{\|\phi_n\| \mid n \geq 1\} < \infty$, so that $x \in l^1(\mathbb{N})$.