Compulsory assignment in MAT4410, Fall 2019

The solutions must be submitted by 14:30 on Thursday, October 31, 2019. Try to solve as many exercises as possible, complete solutions of a half of the exercises is enough to pass.

Problem 1. This problem introduces and establishes the main properties of *conditional expectations*, which is a topic some of you have already studied. If you have difficulties solving the following exercises, consult any text on probability theory.

Let $(X, \mathcal{B}, \mu)$ be a measure space, $\mu(X) < \infty$, and $\mathcal{F} \subset \mathcal{B}$ be a σ-subalgebra.

(i) For every $f \in L^1(X, \mathcal{B}, d\mu)$, consider the complex measure μ_f on $\mathcal{B}$ defined by $\mu_f(A) = \int_A f\, d\mu$. Show that $\mu_f|_{\mathcal{F}}$ is absolutely continuous with respect to $\mu|_{\mathcal{F}}$. Denote by $\mathbb{E}(f|\mathcal{F}) \in L^1(X, \mathcal{F}, d\mu)$ the Radon–Nikodym derivative $\dfrac{d(\mu_f|_{\mathcal{F}})}{d(\mu|_{\mathcal{F}})}$.

The map

$$\mathbb{E}(\cdot|\mathcal{F})\colon L^1(X, \mathcal{B}, d\mu) \to L^1(X, \mathcal{F}, d\mu)$$

is called the conditional expectation.

(ii) Show that the conditional expectations have the following properties:

(a) $\mathbb{E}(\cdot|\mathcal{F})\colon L^1(X, \mathcal{B}, d\mu) \to L^1(X, \mathcal{F}, d\mu)$ is a linear operator, and $\mathbb{E}(f|\mathcal{F}) = f$ for $f \in L^1(X, \mathcal{F}, d\mu)$;

(b) $\mathbb{E}(\cdot|\mathcal{F})$ is positive, in the sense that $\mathbb{E}(f|\mathcal{F}) \geq 0$ whenever $f \geq 0$.

(iii) Let $\varphi\colon \mathbb{C} \to \mathbb{R}$ be a convex function. You may use as a known fact that then φ is continuous.

(a) Consider the set

$$M = \{(a, b, c) \in \mathbb{Q}^3 \mid \varphi(s + it) \geq as + bt + c \text{ for all } s, t \in \mathbb{R}\}$$

and show that, for all $s, t \in \mathbb{R}$,

$$\varphi(s + it) = \sup\{as + bt + c \mid (a, b, c) \in M\}.$$

Hint: use the Hahn–Banach separation theorem, with $\{(s, t, x) \in \mathbb{R}^3 \mid x > \varphi(s + it)\}$ as one of the sets.

(b) Show that if $f \in L^1(X, \mathcal{B}, d\mu)$ is such that $\varphi(f) \in L^1(X, \mathcal{B}, d\mu)$, then

$$\varphi\big(\mathbb{E}(f|\mathcal{F})\big) \leq \mathbb{E}(\varphi(f)|\mathcal{F}).$$

This is called *Jensen's inequality for conditional expectations*. Hint: consider first $\varphi(s + it) = as + bt + c$.

Conclude that for all $1 \leq p < \infty$ the conditional expectation defines a contraction

$$L^p(X, \mathcal{B}, d\mu) \to L^p(X, \mathcal{F}, d\mu).$$

(iv) Show that if $f \in L^1(X, \mathcal{B}, d\mu)$ and $g \in L^\infty(X, \mathcal{F}, \mu|_{\mathcal{F}})$, then

$$\mathbb{E}(fg|\mathcal{F}) = \mathbb{E}(f|\mathcal{F})g.$$

Hint: start with $g = \chi_A$ for $A \in \mathcal{F}$.

(v) Show that $\mathbb{E}(\cdot|\mathcal{F})$ considered as a map $L^2(X, \mathcal{B}, d\mu) \to L^2(X, \mathcal{F}, d(\mu|_{\mathcal{F}}))$ is the orthogonal projection.

Problem 2. Let λ_n be the Lebesgue measure on $\mathbb{R}^n$. Denote by $\ell_x\colon \mathbb{R}^n \to \mathbb{R}^n$ the translation by x: $\ell_x(y) = x + y$.

(i) Show that for every $f \in L^1(\mathbb{R}^n)$ (as usual, when the measure is not specified, we mean λ_n) the map

$$\mathbb{R}^n \to L^1(\mathbb{R}^n), \quad x \mapsto f \circ \ell_x,$$

is continuous. Hint: start with $f \in C_c(\mathbb{R}^n)$.

(ii) The above property has some useful and curious consequences. For example: show that if $A \subset \mathbb{R}^n$ is a Borel set of finite nonzero measure, then the set

$$A - A = \{x - y \mid x, y \in A\}$$

contains a neighbourhood of 0. Hint: what can you say about the integral $\int_{\mathbb{R}^n} \chi_A(x)\chi_A(x+t)d\lambda_n(x)$ as a function of $t \in \mathbb{R}^n$?

(iii) Assume A and B are Borel sets of finite nonzero measure. Show that there are a Borel subset $C \subset A$ of nonzero measure and $x \in \mathbb{R}^n$ such that $x + C \subset B$.

(iv) With A and B as above, assume also that $\lambda_n(A) = \lambda_n(B)$. Show using Zorn's lemma that there exist disjoint Borel subsets $C_k \subset A$ ($k \in \mathbb{N}$) and points $x_k \in \mathbb{R}^n$ such that $\lambda(A \setminus \cup_{k=1}^{\infty} C_k) = 0$, the sets $x_k + C_k$ are disjoint, contained in B, and

$$\lambda_n\big(B \setminus \cup_{k=1}^{\infty}(x_k + C_k)\big) = 0.$$

(v) Extend the previous result to A and B such that $\lambda_n(A) = \lambda_n(B) = +\infty$.

Problem 3. Let G be a locally compact group. Recall (see homework from 4.10.19) that a left Haar measure on G is a nonzero Radon measure λ invariant under the left translations. It is customary to fix such a measure and then write $\int_G f(g)dg$ instead of $\int_G f(g)d\lambda(g)$.

We know that a left Haar measure is unique up to a scalar factor. Since its push-forward under a right translation is still left-invariant (check this!), this push-forward is a scalar multiple of our fixed measure. In other words, for every $h \in G$, there is a scalar $\Delta(h) > 0$ uniquely determined by the property

$$\int_G f(gh)dg = \Delta(h)^{-1} \int_G f(g)dg \quad \text{for all} \quad f \in C_c(G).$$

The map $\Delta\colon G \to (0, +\infty)$ is called the *modular function* of G. If $\Delta(g) = 1$ for all g, then G is called *unimodular.*

(i) Show that $\Delta\colon G \to \mathbb{R}_+^\times$ is a continuous homomorphism (so that $\Delta(gh) = \Delta(g)\Delta(h)$).

(ii) Consider the group $GL_n(\mathbb{R})$ of invertible n-by-n matrices, $n \geq 1$. Show that it is an open subset of the space $\mathrm{Mat}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ of n-by-n matrices and, being equipped with the relative topology, $GL_n(\mathbb{R})$ becomes a locally compact group. Show next that, writing an element $A \in GL_n(\mathbb{R})$ as a matrix $(a_{ij})_{i,j}$, we can define a left Haar measure by

$$dA = |\det(A)|^{-n} \prod_{i,j} da_{ij}.$$

(In other words, denoting by λ_{n^2} the Lebesgue measure on $\mathrm{Mat}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, we define λ by $\frac{d\lambda}{d\lambda_{n^2}}(A) = |\det(A)|^{-n}$.) Show that this measure is also right-invariant, so that the group $GL_n(\mathbb{R})$ is unimodular.

(iii) Consider the closed subgroup $G \subset GL_2(\mathbb{R})$ of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \quad \text{with } a > 0,\ b \in \mathbb{R}.$$

Find a left Haar measure on it and compute the modular function.