Solutions to the exam in MAT4410, Fall 2019

Problem 1

(a) Formulate the Uniform Boundedness Principle.

(b) Let X be a locally compact space. Assume we are given functions f_n ($n \in \mathbb{N}$) and f in $C_0(X)$ such that

$$\int_X f_n \, d\mu \underset{n}{\to} \int_X f \, d\mu \tag{1}$$

for all regular complex Borel measures μ on X. Show that $\sup_n \|f_n\| < \infty$ (where $\|f_n\|$ denotes the supremum-norm of f_n) and $f_n \to f$ pointwise.

(c) Show that conversely, if $f_n, f \in C_0(X)$ are such that $\sup_n \|f_n\| < \infty$ and $f_n \to f$ pointwise, then we have (1) for all complex Borel measures μ on X.

Solution:

(a) Assume $\{T_i \colon X \to Y\}_{i\in I}$ is a collection of bounded linear operators between a Banach space X and a normed space Y such that $\sup_i \|T_i x\| < \infty$ for every $x \in X$. Then $\sup_i \|T_i\| < \infty$.

(b) By a corollary of the Riesz–Markov theorem, the dual space $C_0(X)^*$ can be identified with the space $M(X)$ of regular complex Borel measures on X. Therefore by assumption we have $\varphi(f_n) \to \varphi(f)$ for all $\varphi \in C_0(X)^*$. Hence, by the Uniform Boundedness Principle (applied to the collection of the functions f_n viewed as linear functionals $C_0(X)^* \to \mathbb{C}$), we have $\sup_n \|f_n\| < \infty$. Taking $\mu = \delta_x$ for some $x \in X$, we also get $f_n(x) \to f(x)$, so that $f_n \to f$ pointwise.

(c) For finite positive μ this follows by the Lebesgue dominated convergence theorem. As any complex Borel measure is a linear combination of such measures, we then get the convergence in general.

Problem 2

(a) Formulate the Fubini–Tonelli theorem.

(b) Consider the interval $[0,1]$ and the σ-algebra $\mathcal{B}$ of Borel subsets of $[0,1]$. Let λ and μ be the Lebesgue and counting measures, respectively, on $([0,1],\mathcal{B})$. (Thus $\mu(A)$ equals the number of elements of A.) Denote by D the diagonal $\{(x,x) | x \in [0,1]\}$ in $[0,1]\times[0,1]$ and consider the characteristic function χ_D of D.

Compute the integrals

$$\int_{[0,1]} \left(\int_{[0,1]} \chi_D(x,y) d\lambda(x) \right) d\mu(y) \quad \text{and} \quad \int_{[0,1]} \left(\int_{[0,1]} \chi_D(x,y) d\mu(y) \right) d\lambda(x).$$

Why the Fubini–Tonelli theorem does not apply in this case?

Solution:

(a) A short accepted formulation of the Fubini–Tonelli theorem is that if $(X_1,\mathcal{B}_1,\mu_1)$ and $(X_2,\mathcal{B}_2,\mu_2)$ are σ-finite measure spaces, f is a $(\mathcal{B}_1\times\mathcal{B}_2)$-measurable function on $X_1\times X_2$ which is either nonnegative or integrable with respect to $\mu_1\times\mu_2$, then the integrals

$$\int_{X_1\times X_2} f \, d(\mu_1\times\mu_2), \quad \int_{X_1} \left(\int_{X_2} f(x_1,x_2) d\mu_2(x_2) \right) d\mu_1(x_1), \quad \int_{X_2} \left(\int_{X_1} f(x_1,x_2) d\mu_1(x_1) \right) d\mu_2(x_2) \tag{2}$$

are well-defined and equal. More pedantically:

Fubini: If $f\colon X_1 \times X_2 \to \mathbb{C}$ is integrable with respect to the measure $\mu_1 \times \mu_2$, then $f(\cdot, x_2) \in L^1(X_1, \mathcal{B}_1, d\mu_1)$ for μ_2-a.e. x_2, $f(x_1, \cdot) \in L^1(X_2, \mathcal{B}_2, d\mu_2)$ for μ_1-a.e. x_1, the μ_2-almost everywhere defined function

$$x_2 \mapsto \int_{X_1} f(x_1, x_2) d\mu_1(x_1) \tag{3}$$

is integrable with respect to μ_2, the μ_1-almost everywhere defined function

$$x_1 \mapsto \int_{X_2} f(x_1, x_2) d\mu_2(x_2) \tag{4}$$

is integrable with respect to μ_1, and the integrals (2) are equal.

Tonelli: If $f\colon X_1 \times X_2 \to [0, +\infty]$ is $(\mathcal{B}_1 \times \mathcal{B}_2)$-measurable, then $f(\cdot, x_2)$ is $\mathcal{B}_1$-measurable for all x_2, $f(x_1, \cdot)$ is $\mathcal{B}_2$-measurable for all x_1, the function (3) is $\mathcal{B}_2$-measurable, the function (4) is $\mathcal{B}_1$-measurable, and the integrals (2) are equal (but possibly infinite).

(b) We have $\int_{[0,1]} \chi_D(x, y) d\lambda(x) = 0$ for all $y \in [0, 1]$, hence

$$\int_{[0,1]} \left(\int_{[0,1]} \chi_D(x, y) d\lambda(x) \right) d\mu(y) = 0.$$

We also have $\int_{[0,1]} \chi_D(x, y) d\mu(y) = 1$ for all $x \in [0, 1]$, hence

$$\int_{[0,1]} \left(\int_{[0,1]} \chi_D(x, y) d\mu(y) \right) d\lambda(x) = 1.$$

The Fubini–Tonelli theorem does not apply in this case, since μ is not σ-finite.

Problem 3

(a) Formulate the Radon–Nikodym theorem.

(b) Consider the measures λ and μ from Problem 2b. Show that $\lambda \ll \mu$, but there is no Radon–Nikodym derivative $\frac{d\lambda}{d\mu}$. What goes wrong with the Radon–Nikodym theorem here?

(c) Show that there is no Lebesgue decomposition of μ with respect to λ, that is, we cannot write $\mu = \mu_a + \mu_s$, with $\mu_a \ll \lambda$ and $\mu_s \perp \lambda$.

Solution:

(a) Assume ν and μ are σ-finite measures on a measurable space $(X, \mathcal{B})$, $\nu \ll \mu$. Then there is a $\mathcal{B}$-measurable function $f\colon X \to [0, +\infty)$ such that

$$\nu(A) = \int_A f\, d\mu \quad \text{for all} \quad A \in \mathcal{B}.$$

The function f is essentially unique, in the sense that if $\tilde{f}$ is another such function, then $f = \tilde{f}$ μ-a.e.

(b) We have to show that if $\mu(A) = 0$ for a Borel set A, then $\lambda(A) = 0$. This is obviously the case, as $\mu(A) = 0$ only for $A = \emptyset$.

Assume f is the Radon–Nikodym derivative $\frac{d\lambda}{d\mu}$, so that

$$\lambda(A) = \int_A f\, d\mu = \sum_{x \in A} f(x) \quad \text{for all} \quad A \in \mathcal{B}.$$

Taking $A = \{x\}$, we conclude that $f(x) = 0$ for all x. This implies that $\lambda = 0$, which is nonsense.

The Radon–Nikodym theorem does not apply in this case, since μ is not σ-finite.

(c) Assume there is a Lebesgue decomposition $\mu = \mu_a + \mu_s$. As $\mu_s \perp \lambda$, by definition there is a Borel set $N \subset [0,1]$ such that $\lambda(N) = 0$ and $\mu_s(N^c) = 0$. Then for every $x \in N^c$ we have $\mu_a(\{x\}) = 0$ (as $\lambda(\{x\}) = 0$) and $\mu_s(\{x\}) = 0$, hence $\mu(\{x\}) = 0$. As μ is the counting measure, this means that $N^c = \emptyset$, so that $N = [0,1]$, which contradicts $\lambda(N) = 0$.

Problem 4

Assume $f\colon \mathbb{R}^n \to \mathbb{C}$ is an integrable function and consider its Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi x} dx,$$

where ξx denotes the scalar product $\xi_1 x_1 + \cdots + \xi_n x_n$. For $k \in \mathbb{N}$, consider also the function ϕ_k on $\mathbb{R}^n$ defined by

$$\phi_k(x) = k^n e^{-\pi k^2 x^2}.$$

(a) Show that the functions $\phi_k * f$ are continuous and for every point of continuity x of f we have

$$(\phi_k * f)(x) \underset{k}{\to} f(x).$$

(b) Show that if f is continuous at 0 and $\hat{f} \geq 0$, then $\hat{f}$ is integrable and

$$\int_{\mathbb{R}^n} \hat{f}(\xi) d\xi = f(0).$$

Solution:

It is important to remember that $\int \phi_1(x)dx = 1$ and $\hat{\phi}_1(\xi) = \phi_1(\xi)$. Hence $\int \phi_k(x)dx = 1$ and $\hat{\phi}_k(\xi) = e^{-\pi \xi^2/k^2}$.

(a) For the first claim, since the functions ϕ_k are uniformly continuous, the same proof as for compactly supported continuous functions that we had in the class works. Namely, with k fixed, take $\varepsilon > 0$ and choose $\delta > 0$ such that $|\phi_k(x) - \phi_k(y)| < \varepsilon$ whenever $|x - y| < \delta$ (where $|x - y|$ denotes the Euclidean norm). Then, if $|x - y| < \delta$, we have

$$\begin{aligned} |(\phi_k * f)(x) - (\phi_k * f)(y)| &= \left| \int \big(\phi_k(x-z) - \phi_k(y-z)\big) f(z) dz \right| \\ &\leq \int |\phi_k(x-z) - \phi_k(y-z)|\, |f(z)| dz \\ &\leq \varepsilon \int |f(z)| dz = \varepsilon \|f\|_1. \end{aligned}$$

This shows that $\phi_k * f$ is continuous, and even uniformly continuous.

Now assume that x is a point of continuity of f. Fix $\varepsilon > 0$ and choose $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ whenever $|y - x| < \delta$. As $\int \phi_k(z)dz = 1$, we have

$$\begin{aligned} |(\phi_k * f)(x) - f(x)| &= \left| \int \phi_k(z) f(x-z) dz - f(x) \int \phi_k(z) dz \right| \\ &\leq \int_{B_\delta(0)} \phi_k(z) |f(x-z) - f(x)| dz + \int_{B_\delta(0)^c} \phi_k(z) |f(x-z)| dz + |f(x)| \int_{B_\delta(0)^c} \phi_k(z) dz. \end{aligned}$$

The first summand above is $< \varepsilon$ by our choice of δ. Let us look at the other two summands for large k.

The second summand is not larger than

$$\int_{B_\delta(0)^c} |f(x-z)| dz \cdot \sup_{z \in B_\delta(0)^c} \phi_k(z) \leq \|f\|_1 k^n e^{-\pi \delta^2 k^2},$$

which converges to 0 as $k \to \infty$.

The third summand equals

$$|f(x)| \int_{B_{k\delta}(0)^c} e^{-\pi z^2} dz,$$

which converges to 0 as $k \to \infty$, since the function $e^{-\pi z^2}$ is integrable.

To summarize, we have

$$\limsup_k |(\phi_k * f)(x) - f(x)| \leq \varepsilon,$$

proving the second claim in (a), as $\varepsilon > 0$ was arbitrary.

(b) As $\hat{\phi}_k(\xi) = e^{-\pi\xi^2/k^2}$ is integrable and $\widehat{\phi_k * f} = \hat{\phi}_k \hat{f}$, the function $\widehat{\phi_k * f}$ is integrable. Therefore by the Fourier inversion formula we have

$$(\phi_k * f)(x) = \int_{\mathbb{R}^n} \widehat{\phi_k * f}(\xi) e^{2\pi i x\xi} d\xi = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{-\pi\xi^2/k^2} e^{2\pi i x\xi} d\xi$$

for almost all x. But as both sides of this equality are continuous in x (the left hand side - by part (a), the right hand side - because the Fourier transform of an integrable function is always continuous), we conclude that the equality holds for all x. (As a side note, this is what we also established in the proof of the Fourier inversion theorem in the class.) In particular,

$$(\phi_k * f)(0) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{-\pi\xi^2/k^2} d\xi.$$

Letting $k \to \infty$, the left hand side converges to $f(0)$ by (a), while the right hand side converges to $\int \hat{f}\, d\xi$ by the monotone convergence theorem. Hence

$$f(0) = \int_{\mathbb{R}^n} \hat{f}(\xi) d\xi.$$