# MAT4410 (2020 AUTUMN) MANDATORY ASSIGNMENT 2

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This is a second set of mandatory assignment, just for those who did not pass the first one. You need to upload your solution to the Canvas system, by November 22 (Sunday). Solve at least one question in each section.

## 1. Measure theory

**Problem 1.** Let $X_0 = \{0, 1\}$ and $\nu_0$ be the probability measure on $X_0$ characterized by $\nu_0(\{0\}) = \frac{1}{4}$. We then put $X = \prod_{i=1}^{\infty} X_0$, and $\nu = \bigotimes_{i=1}^{\infty} \nu_0$. As usual, the domain of $\nu$ is the $\sigma$-algebra generated by subsets $A \subset X$ of the form $A = A' \times \prod_{i=k+1}^{\infty} X_0$ for $A' \subset \prod_{i=1}^{k} X_0$ for $k \in \mathbb{N}$. Show that $(X, \mathcal{M}, \nu)$ is isomorphic to $([0, 1], \mathcal{B}_{[0,1]}, m)$, where $m$ is the Lebesgue measure on $[0, 1]$.

**Problem 2.** Let $S$ be an uncountable infinite set, and let $X$ be the union of $S$-many copies of $\mathbb{R}$. Formally speaking, we can treat $S$ as a topological space with the discrete topology, and take $X = \mathbb{R} \times S$ with the product topology. With this convention, consider the functional $\phi$ on $C_c(X)$ defined by

$$\phi(f) = \sum_{s \in S} \int f(t, s) dt.$$

(In other words, it is the sum of integrable by the Lebesgue measure on each copy of $\mathbb{R}$ in $X$, which are $S$-many of them.) Let $\mu$ be the corresponding Radon measure on $X$.

(1) Let $E \subset X$ a measurable set, such that $E^s = \{t \in \mathbb{R} \mid (t, s) \in E\}$ is not empty for infinitely many $s \in S$. Show that $\mu(E) = \infty$.
(2) Explain that $\mu$ is not inner regular for $E = \{0\} \times S \subset X$.

## 2. $L^p$ and Banach spaces

**Problem 3.** Let $(X, \mathcal{M})$ be a measurable space. When $\mu$ is a complex measure on $(X, \mu)$, we denote its total variation by $|\mu|$. (See Problem 7 from Exercise Set 4, or lecture note from September 22.) We denote the space of complex measures on $(X, \mu)$ by $M(X)$.

(1) Check that $M(X)$ is a complex vector space.
(2) Show that $\|\mu\| = |\mu|(X)$ defines a norm on $M(X)$.
(3) Show that $M(X)$ is complete for this norm.

**Problem 4.** Let $X$ be a normed vector space.

(1) Let $x_1, \ldots, x_k$ be a finite collection of vectors in $X$. Show that, if $k < \dim X$, then there is a vector $x' \in X$ such that $\|x'\| = 1$ and $\|x' - x_i\| > \frac{1}{2}$ for $1 \leq i \leq k$.
Hint: let $Y$ be the subspace spanned by the vectors $x_i$. When you consider the quotient space $Y/X$, is there a unit vector for the quotient seminorm?
(2) Show that, if $X$ is infinite dimensional, the unit ball $\{x \in X \mid \|x\| \leq 1\}$ of $X$ is not compact.

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